A New Mathematical Programming Framework for Facility Layout

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The Facility Layout (or Floorplanning) Problem

- Find the optimal positions for a given set of N rectangular departments of *fixed area* within a rectangular facility of *fixed area*.
- All the dimensions may be given <u>or</u> left undetermined.
- The objective is to minimize (according to some norm, e.g. l_1, l_2) the distances between pairs of departments that have a nonzero connection "cost".

Applications ?

- Hospital layout
- Service center layout
- VLSI placement and design
- etc.

But : Like many optimization problems from practical applications, the facility layout problem is “hard” (NP-hard).

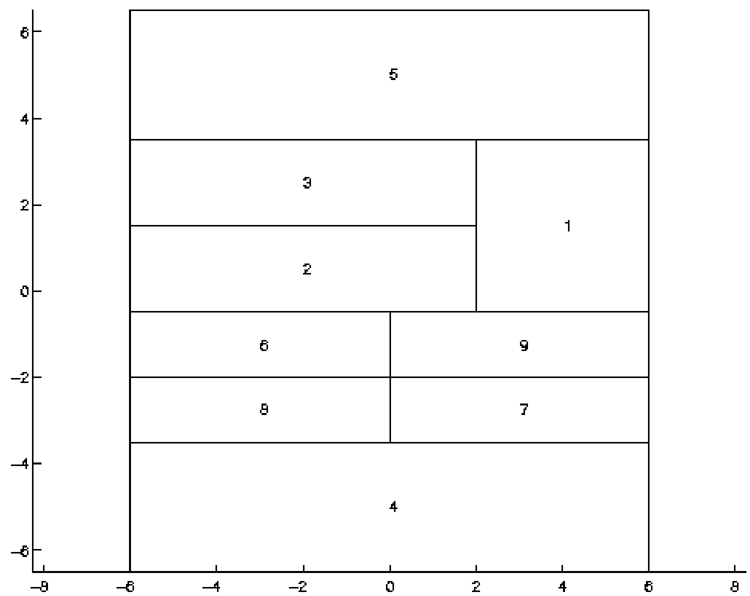
Small Example

Dept	Area
1	16
2	16
3	16
4	36
5	36
6	9
7	9
8	9
9	9

i	***j***	C_{ij}	***i***	***j***	C_{ij}	***i***	***j***	C_{ij}
1	4	5	2	9	1	4	7	4
1	5	5	3	4	2	5	6	3
1	9	1	3	5	2	5	9	4
2	4	3	3	9	1	6	9	2
2	5	3	4	6	4	7	9	1

Facility is fixed to be a 12 x 13 rectangle

Possible Layout for Example



Total Euclidean cost: 229.71

Motivation for this Work

- Exact mixed integer programming approaches only work for problems with less than 10 departments.

- Most other approaches in the literature are based on heuristic search methods.

➡ We present a new two-stage framework based on mathematical programming models, and inspired by a *convex global relaxation* of the layout problem.

Outline of the Proposed New Framework

✓ The first model is a *convex relaxation* of the layout problem (to find a good starting point);

✓ The second model is an *exact formulation* of the problem as a mathematical program with equilibrium constraints (MPEC).

➔ Both models can be *solved efficiently* using widely available non-linear optimization software.

The NLT method

(van Camp, Carter & Vannelli, 1991)

The (non-convex) vCCV model

$$\min_{(x_i, y_i), h_i, w_i, h_F, w_F} \sum_{1 \le i < j \le N} c_{ij} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

$$s.t. \quad \begin{cases} |x_i - x_j| - \frac{1}{2}(w_i + w_j) \ge 0 \quad \text{if} \quad |y_i - y_j| - \frac{1}{2}(h_i + h_j) < 0, \ \forall i \ne j, \\ |y_i - y_j| - \frac{1}{2}(h_i + h_j) \ge 0 \quad \text{if} \quad |x_i - x_j| - \frac{1}{2}(w_i + w_j) < 0, \ \forall i \ne j, \end{cases}$$

$$\begin{cases} \frac{1}{2} w_F - (x_i + \frac{1}{2} w_i) \ge 0 \ \ \forall i, & \frac{1}{2} h_F - (y_i + \frac{1}{2} h_i) \ge 0 \ \ \forall i, \\ (x_i - \frac{1}{2} w_i) + \frac{1}{2} w_F \ge 0 \ \ \forall i, & (y_i - \frac{1}{2} h_i) + \frac{1}{2} h_F \ge 0 \ \ \forall i, \end{cases}$$

$$\begin{cases} \min(w_i, h_i) - l_i^{\min} \ge 0 \ \ \forall i, & l_i^{\max} - \min(w_i, h_i) \ge 0 \ \ \forall i, \\ \min(w_F, h_F) - l_F^{\min} \ge 0, & l_F^{\max} - \min(w_F, h_F) \ge 0, \end{cases}$$

$$h_i w_i = a_i \quad \forall i$$

where $l_i^{\min}$, $l_i^{\max}$, $l_F^{\min}$ and $l_F^{\max}$ are given bounds on the dimensions.

NLT: Three-Stage Approach

(1) Evenly distribute the centres of the departments inside the facility;

(2) Reduce the overlap between departments;

(3) Determine the final solution by solving the vCCV model.

Stages 1 and 2 are (non-convex) relaxations of the vCCV model which approximate the departments by circles.

The Stage-2 model of NLT

$$\min_{(x_i, y_i), h_F, w_F} \sum_{1 \le i < j \le N} c_{ij}\, d_{ij}$$

subject to

$$d_{ij} \ge r_i + r_j \quad \forall i \ne j$$

$$\tfrac{1}{2} w_F \ge x_i + r_i \;, \quad \tfrac{1}{2} w_F \ge r_i - x_i \quad \forall i$$

$$\tfrac{1}{2} h_F \ge y_i + r_i \;, \quad \tfrac{1}{2} h_F \ge r_i - y_i \quad \forall i$$

$$l_F^{\max} \ge \min(w_F, h_F) \ge l_F^{\min},$$

where $d_{ij} := \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

and r_i is the radius of the circle for department i.

ModCoAR:
The First Model of
the New Framework

Steps to derive the ModCoAR model

(1) Define the *target distance* concept for circles with varying radii;

(2) Enforce the target distances using a *repeller* term in the objective function;

(3) Analyse and *convexify* the result

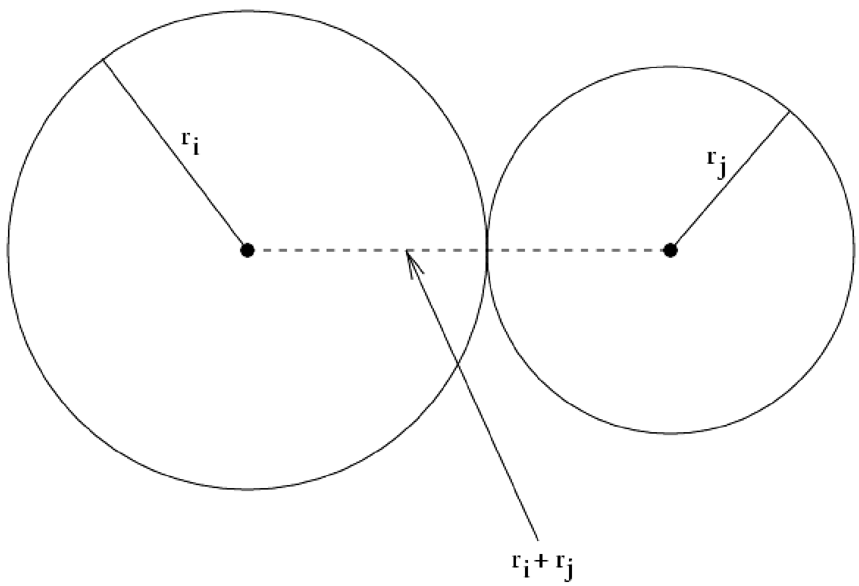
➡ Concept of *generalized target distances*;

(4) Add a *barrier term* (for ease of computation).

For convenience, we work with the squares of the distances:

$$D_{ij} := d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$$

Target distances concept



Hence the target distance between circles *i* and *j* is

$$t_{ij} := \alpha \cdot (r_i + r_j)^2 \quad \text{for some } \alpha > 0.$$

Attractor-Repeller Paradigm

For each pair *i*,*j* of modules, the distance minimizing term is viewed as an attractor:

$$c_{ij} \cdot D_{ij}, \qquad c_{ij} \geq 0, \quad D_{ij} \geq 0$$

is minimized when $D_{ij} = 0$.

Enforcing the target distances

To counter this "attraction", we enforce the target distances with *repeller* terms in the objective function:

$$f(z) := \frac{1}{z} - 1, \quad z > 0$$

and $z = \frac{D_{ij}}{t_{ij}}$, where t_{ij} is the target distance.

The (non-convex) AR model

$$\min_{(x_i, y_i), h_F, w_F} \sum_{1 \le i < j \le N} c_{ij} D_{ij} + \sum_{1 \le i < j \le N} f\left(\frac{D_{ij}}{t_{ij}}\right)$$

subject to ~~$d_{ij} \ge r_i + r_j \quad \forall i \ne j$~~

$$\tfrac{1}{2} w_F \ge x_i + r_i \quad \forall i$$

$$\tfrac{1}{2} h_F \ge y_i + r_i \quad \forall i$$

$$\tfrac{1}{2} w_F \ge r_i - x_i \quad \forall i$$

$$\tfrac{1}{2} h_F \ge r_i - y_i \quad \forall i$$

~~$l_F^{\max} \ge \min(w_F, h_F) \ge l_F^{\min}$~~

$$w_F^{\max} \ge w_F \ge w_F^{\min}$$

$$h_F^{\max} \ge h_F \ge h_F^{\min}$$

Examine the objective function

Rewrite the objective function:

$$\sum_{1 \le i < j \le N} c_{ij} D_{ij} + \sum_{1 \le i < j \le N} \left(\frac{t_{ij}}{D_{ij}} - 1 \right) = \sum_{1 \le i < j \le N} \left(c_{ij} D_{ij} + \frac{t_{ij}}{D_{ij}} - 1 \right)$$

and since the sum of convex functions is convex, we ask:

When is the term $c_{ij} D_{ij} + \dfrac{t_{ij}}{D_{ij}} - 1$ convex?

Fact: Let

$$g : \Re^4 \to \Re, \qquad g(x_1, x_2, y_1, y_2) = c\,z + \frac{t}{z} - 1,$$

where

$$c > 0, t > 0 \text{ and } z > 0,\; z = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

Then the following statements hold for g:

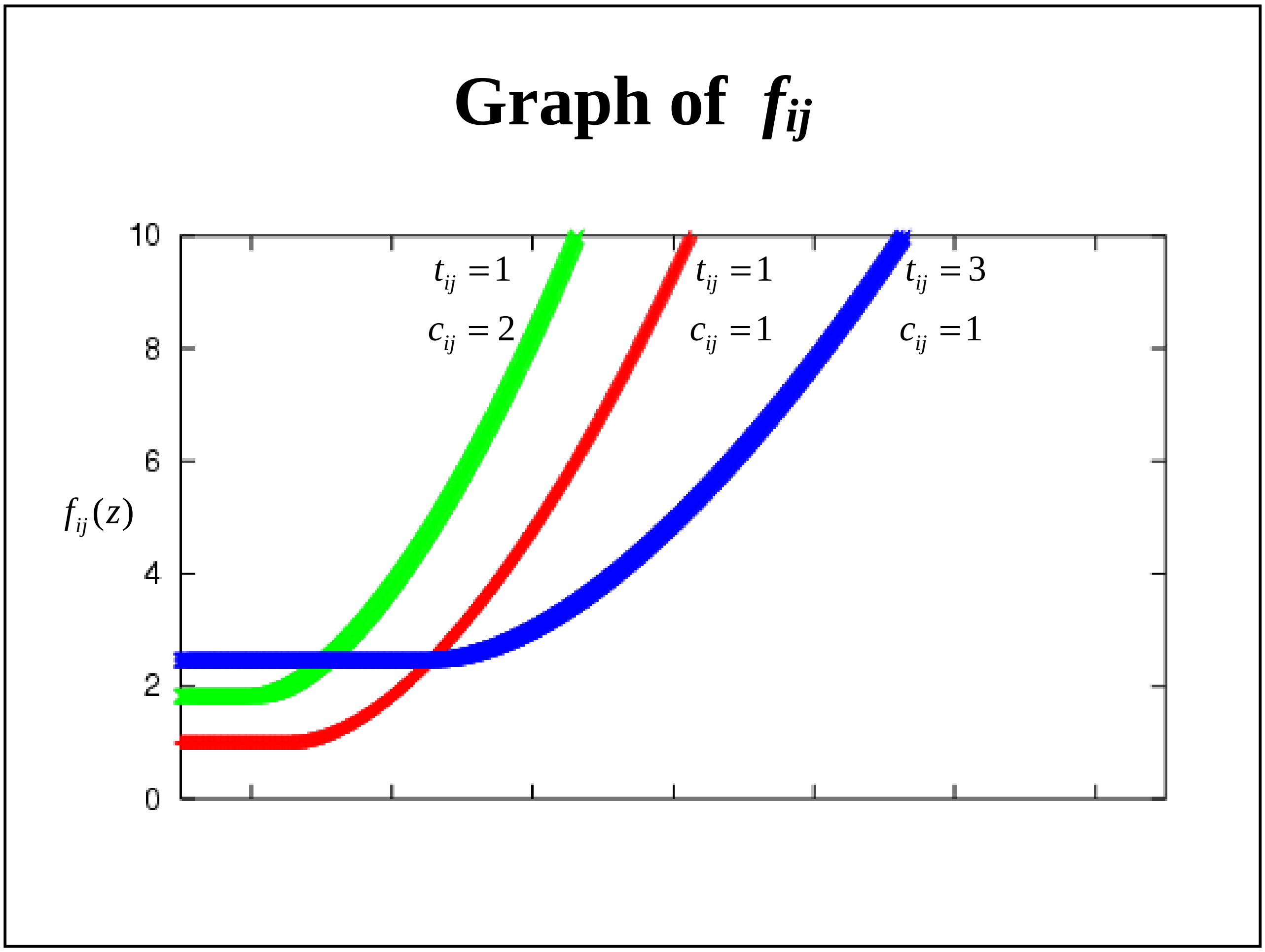
1. If $z \geq \sqrt{\frac{t}{c}}$ then the Hessian of g is positive semidefinite.

2. If $z = \sqrt{\frac{t}{c}}$ then the gradient of g is zero.

Define a new (convex!) function

For $c_{ij} > 0,\ t_{ij} > 0,$ and $D_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2,$ we define the convex, continuously differentiable piecewise function

$$f_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij}\, D_{ij} + \dfrac{t_{ij}}{D_{ij}} - 1, & D_{ij} \geq \sqrt{\dfrac{t_{ij}}{c_{ij}}} \\ 2\sqrt{c_{ij}\ t_{ij}} - 1, & 0 \leq D_{ij} < \sqrt{\dfrac{t_{ij}}{c_{ij}}} \end{cases}$$



Graph of f_{ij}
10
8
6
4
2
0
$f_{ij}(z)$
$t_{ij}=1$
$c_{ij}=2$
$t_{ij}=1$
$c_{ij}=1$
$t_{ij}=3$
$c_{ij}=1$

The *convex* CoAR model

$$\min_{(x_i, y_i),\, h_F, w_F} \quad \sum_{1 \le i < j \le N} f_{ij}(x_i, x_j, y_i, y_j)$$

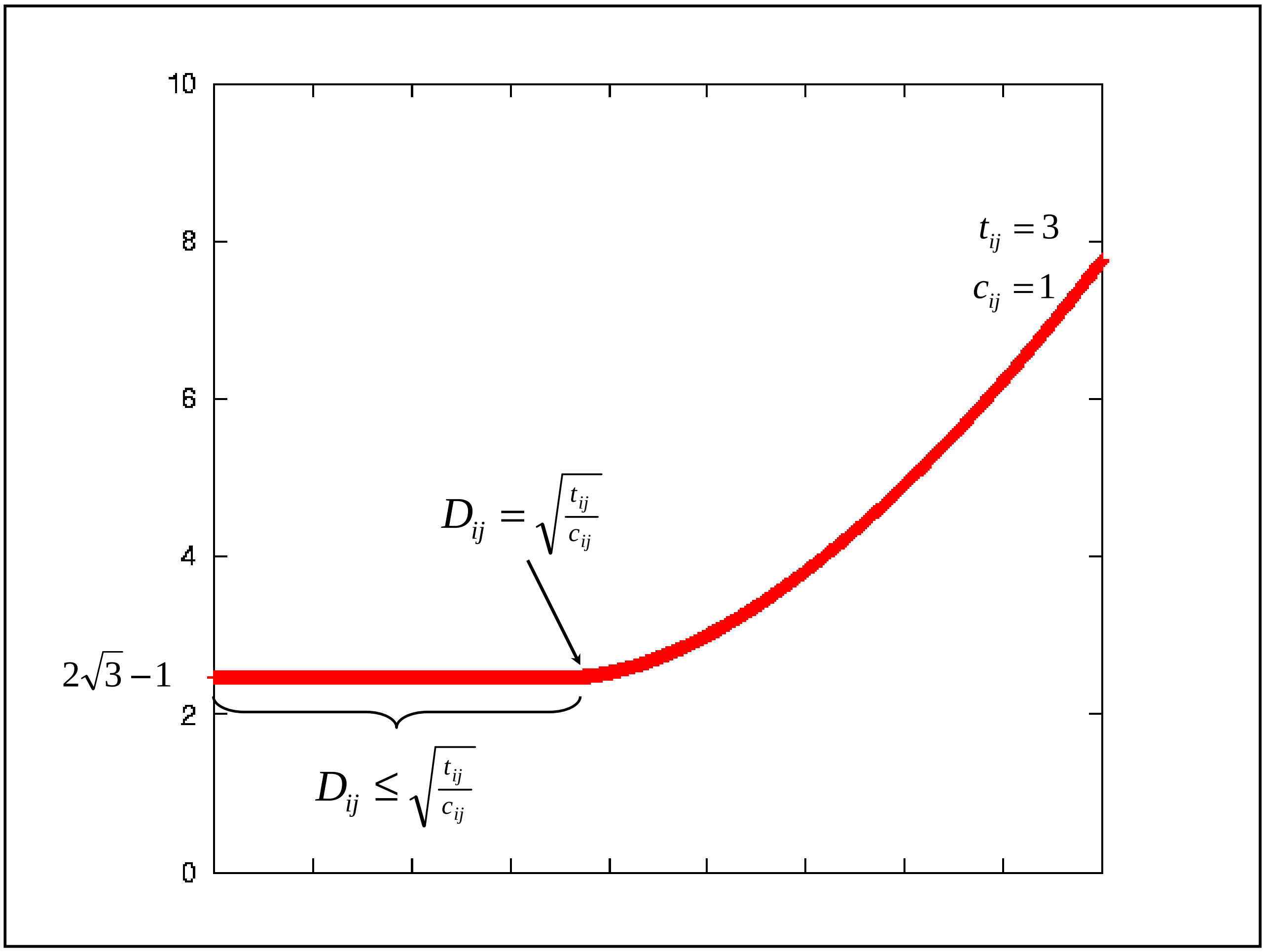
$$\begin{aligned}
\textit{subject to} \quad & \tfrac{1}{2} w_F \ge x_i + r_i \quad \forall i \in M \\
& \tfrac{1}{2} h_F \ge y_i + r_i \quad \forall i \in M \\
& \tfrac{1}{2} w_F \ge r_i - x_i \quad \forall i \in M \\
& \tfrac{1}{2} h_F \ge r_i - y_i \quad \forall i \in M \\
& w_F^{\max} \ge w_F \ge w_F^{\min} \\
& h_F^{\max} \ge h_F \ge h_F^{\min}
\end{aligned}$$

When is the value of f_{ij} minimum?

- We deduce from the structure of f_{ij} that its minimum value is attained for all positions of the departments i and j for which

$$D_{ij} \leq \sqrt{\frac{t_{ij}}{c_{ij}}}.$$

- This includes $D_{ij}=0$ (complete overlap!).



10
8
6
4
2
0
$t_{ij} = 3$
$c_{ij} = 1$
$D_{ij} = \sqrt{\frac{t_{ij}}{c_{ij}}}$
$2\sqrt{3} - 1$
$D_{ij} \leq \sqrt{\frac{t_{ij}}{c_{ij}}}$

Since we want to minimize overlap, what we really want is a layout for which

$$D_{ij} \approx \sqrt{\frac{t_{ij}}{c_{ij}}}.$$

For such a point, we have

D_{ij} proportional to t_{ij} ;

hence our original target distances are still enforced.

Generalized Target Distances

If we define

$$T_{ij} := \sqrt{\frac{t_{ij}}{c_{ij} + \varepsilon}}, \qquad \varepsilon > 0 \quad \text{"small"}$$

then we can think of T_{ij} as a

generalized target distance

for the departments i and j.

This “new” target distance takes <u>both</u> t_{ij} and c_{ij} into account.

Practical Interpretation of T_{ij}

$$T_{ij} := \sqrt{\frac{t_{ij}}{c_{ij}+\varepsilon}}$$

- If c_{ij} is small, then departments i and j are likely to be placed far apart in the layout, so the corresponding T_{ij} can be large;
- If c_{ij} is large, then the opposite reasoning applies, and T_{ij} can be small;
- But T_{ij} also takes t_{ij} into account!

How to achieve T_{ij} ?

Add to the objective function a term of the form

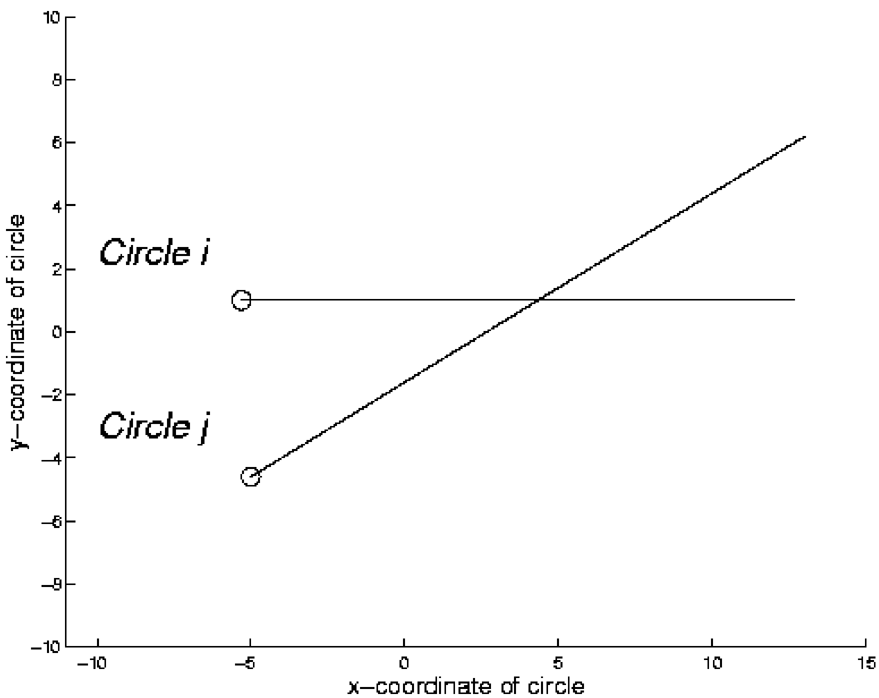
$$-K \ln\left(\frac{D_{ij}}{T_{ij}}\right)$$

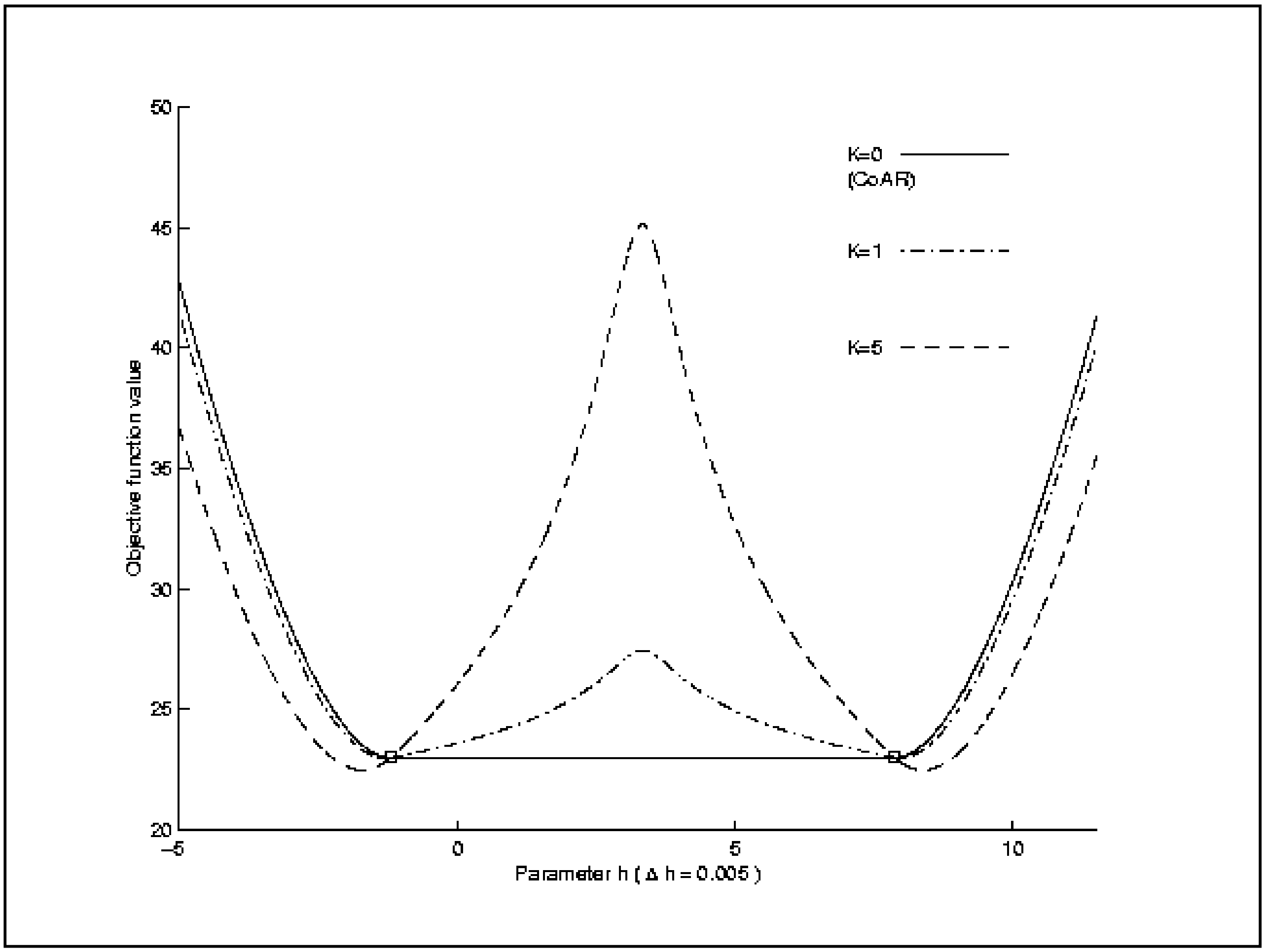
for each pair of departments.

The resulting function has minima that satisfy

$$D_{ij} \approx T_{ij}$$

A Glimpse of the 4D Function…





50
45
40
35
30
25
20
–5
0
5
10
Objective function value
Parameter h (Δ h = 0.005)
K=0
(CoAR)
K=1
K=5

The ModCoAR model

$$\min_{(x_i, y_i),\, h_F, w_F} \quad \sum_{1 \le i < j \le N} F_{ij}(x_i, x_j, y_i, y_j) - K \ln(D_{ij} / T_{ij})$$

subject to

$$\tfrac{1}{2} w_F \ge x_i + r_i, \quad \tfrac{1}{2} h_F \ge y_i + r_i \quad \forall i$$

$$\tfrac{1}{2} w_F \ge r_i - x_i, \quad \tfrac{1}{2} h_F \ge r_i - y_i \quad \forall i$$

$$w_F^{\max} \ge w_F \ge w_F^{\min}, \quad h_F^{\max} \ge h_F \ge h_F^{\min}$$

where

$$F_{ij}(x_i, x_j, y_i, y_j) := \begin{cases} c_{ij} D_{ij} + \dfrac{t_{ij}}{D_{ij}} - 1, & D_{ij} \ge T_{ij} \\ 2\sqrt{c_{ij}\, t_{ij}} - 1, & 0 \le D_{ij} < T_{ij} \end{cases}$$

BPL :
The Second Model of
the New Framework

Non-overlap constraints

$$|x_i - x_j| - \tfrac{1}{2}(w_i + w_j) \geq 0 \quad \text{if} \quad |y_i - y_j| - \tfrac{1}{2}(h_i + h_j) < 0$$

$$|y_i - y_j| - \tfrac{1}{2}(h_i + h_j) \geq 0 \quad \text{if} \quad |x_i - x_j| - \tfrac{1}{2}(w_i + w_j) < 0$$

Note that these constraints are *disjunctive*…

… and therefore can be written as

$$\tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right| \le 0 \quad \text{or} \quad \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right| \le 0$$

which is equivalent to

$$\min \left\{ \tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right|, \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right| \right\} \le 0$$

New variables

For each pair of departments, introduce two new variables

$$X_{ij} \;, Y_{ij}$$

and let

$$X_{ij} \geq \tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right| \quad , \quad X_{ij} \geq 0$$

$$Y_{ij} \geq \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right| \quad , \quad Y_{ij} \geq 0$$

Equilibrium Constraints

Then

$$\min\left\{\tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right|, \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right|\right\} \le 0$$

is equivalent to

$$X_{ij}\, Y_{ij} = 0$$

MPEC Formulation

(Math. Prog. With Equilibrium Constraints)

$$\min_{(x_i, y_i), h_i, w_i, h_F, w_F} \quad \sum_{1 \le i < j \le N} c_{ij} \, \delta_{ij}$$

s. t. $\quad X_{ij} \ge \tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right| \quad , \quad Y_{ij} \ge \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right|$

$$X_{ij} \, Y_{ij} = 0 \quad , \quad X_{ij} \ge 0 \quad , \quad Y_{ij} \ge 0 \quad , \quad \forall \, 1 \le i < j \le N$$

$h_i \, w_i = a_i \quad \forall \, i$ (area constraints)

plus: “fit-in-the-facility” constraints and bound constraints (all linear)

and $\delta_{ij}(x_i, x_j, y_i, y_j)$ is the desired norm (l_1, l_2, …).

Computational Results

Solution methodology

- We solve both models using the software package MINOS.

- For ModCoAR, because of the linearity of the constraints, convergence is generally *superlinear*.

Solution methodology (ctd)

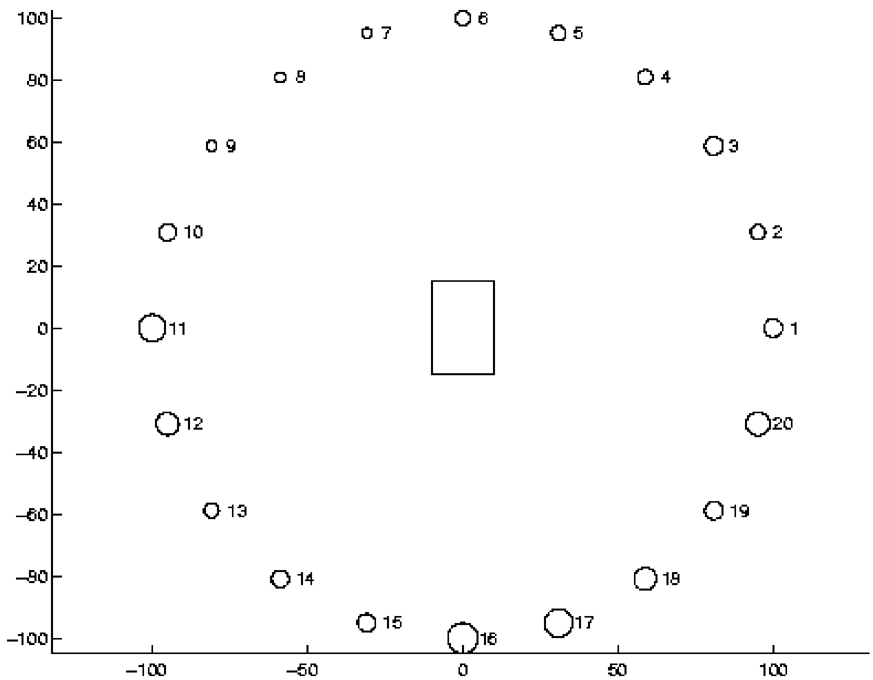
- We chose to set

$$K = \sum_{1 \le i < j \le N} c_{ij}$$

so that K clearly dominates the c_{ij}'s.

- MINOS requires an initial configuration to start the (iterative) algorithm for solving ModCoAR.

Choice of initial configuration



Solving the MPEC using MINOS

- The complementarity constraints

$$X_{ij}\, Y_{ij} = 0 \quad , \quad X_{ij} \geq 0 \quad , \quad Y_{ij} \geq 0$$

  imply that **no** strictly feasible point exists.
  This causes MINOS to fail…

- Thus we apply a penalty-type approach to the above constraints.

BPL Model

$$\min_{(x_i, y_i), h_i, w_i, h_F, w_F} \sum_{1 \le i < j \le N} c_{ij}\, \delta_{ij} \quad + \quad K \cdot X_{ij}\, Y_{ij}$$

$$\text{s. t.} \quad X_{ij} \ge \tfrac{1}{2}(w_i + w_j) - \left| x_i - x_j \right| \quad , \quad Y_{ij} \ge \tfrac{1}{2}(h_i + h_j) - \left| y_i - y_j \right|$$

$$X_{ij} \ge 0 \quad , \quad Y_{ij} \ge 0 \quad , \quad \forall\, 1 \le i < j \le N$$

$$h_i\, w_i = a_i \quad \forall\, i$$

plus: "fit-in-the-facility" constraints and bound constraints (all linear)

Aspect ratio constraints

- The aspect-ratio for department *i* is defined as

$$\beta_i := \frac{\max\{h_i, w_i\}}{\min\{h_i, w_i\}}$$

- Bounding (above) the aspect ratio ensures that no departments are excessively narrow in the layout.

Aspect ratio constraints (ctd)

- We enforce the bound β_i^* on β_i by adding to BPL the constraints

$$\beta_i\, w_i \geq h_i\ , \quad \beta_i\, h_i \geq w_i\ , \quad \beta_i^* \geq \beta_i\ .$$

Classical example : Armour & Buffa problem (1963)

- Large problem (20 departments) – beyond all previous mathematical programming approaches (mixed integer programming).

- Each run of our algorithm requires approximately 18 seconds of CPU time (300 MHz SunSPARC).

- We can compare our framework using the rectilinear norm with the most recent results in the literature (Tate & Smith’95 -- genetic algorithm).

Experiments with the Armour & Buffa problem (1)

First we set no aspect ratio constraints, only a lower bound of 2 on all heights and widths.

We found a layout with cost 4230.6

and aspect ratio 6.67

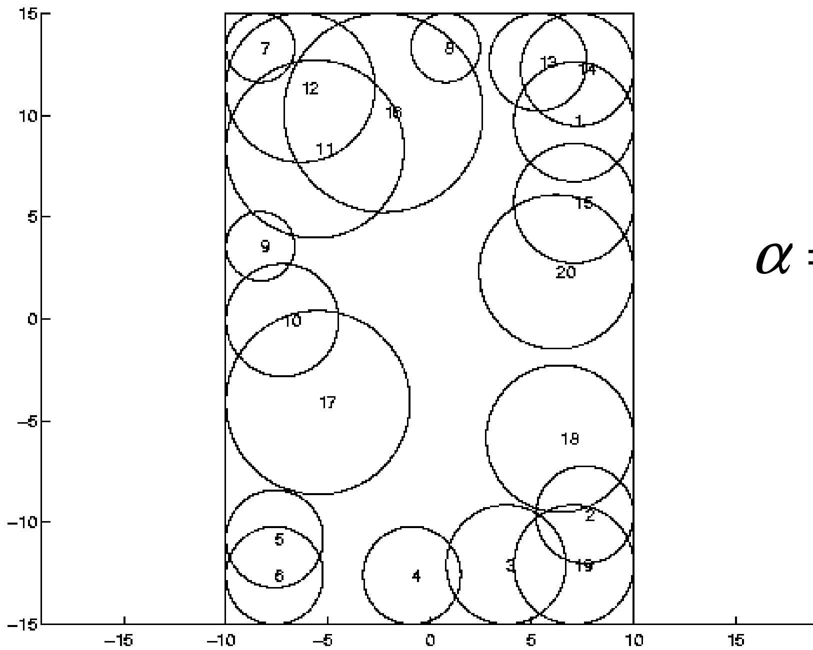
In TS'95, the best layout with aspect ratio bounded by 7 has cost 5255.0

Experiments with the Armour & Buffa problem (2)

Then we started setting aspect ratio constraints:

β_i^*	TS'95	New framework
5	5524.7	4591.3
4	5743.1	4786.4
3	5832.6	5140.1
2	6171.1	5224.7

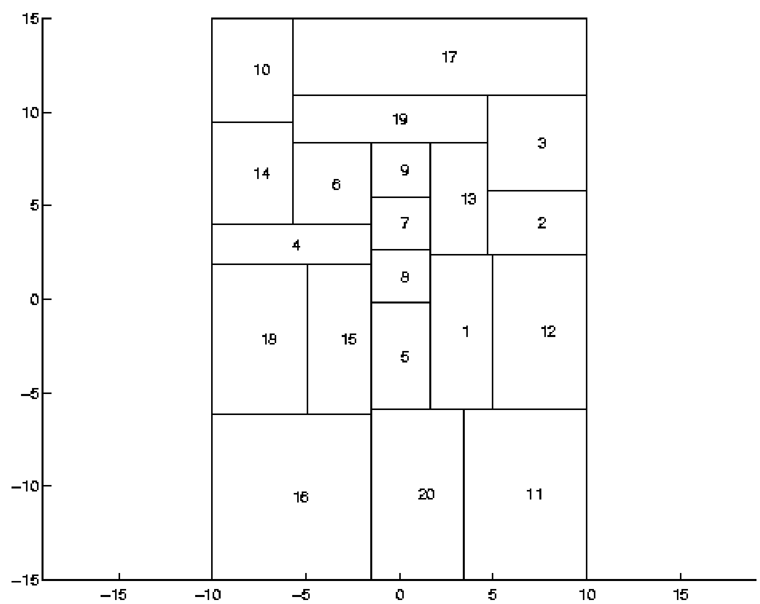
Best layout with $\beta_i \leq 4$



$\alpha = 1.532$

Optimal solution of ModCOAR

Best layout with $\beta_i \leq 4$ (ctd)



Total cost 4786.4 (versus 5743.1 in TS'95)

On-going and Future Research

- Apply this framework to the MCNC macro-cell layout problems, and compare the results with other methods.
- Investigate more thoroughly the role of α in the model.
- Improve the solution methodology; in particular, apply a nonlinear programming solver that directly tackles the MPEC formulation in spite of the lack of a strictly feasible point.

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