Superrigidity and classification problems for torsion-free abelian groups of finite rank

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Definition 1 *$G \leqslant \mathbb{Q}^n$ has rank n iff G contains n linearly independent elements.*

Definition 2 *$R(\mathbb{Q}^n)$ is the standard Borel space of all subgroups $G \leqslant \mathbb{Q}^n$ of rank n.*

Definition 3 *$\cong_n$ denotes the isomorphism relation on $R(\mathbb{Q}^n)$.*

Theorem 4 (S.T. 2000) *$(\cong_n) <_B (\cong_{n+1})$ for all $n \geq 1$.*

Definition 5 *An abelian group A is said to be p-local iff A is q-divisible for every prime $q \neq p$.*

Definition 6 *$R^{(p)}(\mathbb{Q}^n)$ is the standard Borel space of all p-local subgroups $G \leqslant \mathbb{Q}^n$ of rank n.*

Definition 7 *$\cong_n^{(p)}$ denotes the isomorphism relation on $R^{(p)}(\mathbb{Q}^n)$.*

Theorem 8 (S.T. 2000) *Fix a prime p. Then $(\cong_n^{(p)}) <_B (\cong_{n+1}^{(p)})$ for all $n \geq 1$.*

Question 9 *Fix some prime p and some $n \geq 2$. Is the classification problem for the p-local torsion-free abelian groups of rank n strictly easier than that for* arbitrary *torsion-free abelian groups of rank n?*

Question 10 *Fix some $n \geq 2$ and let $p \neq q$ be distinct primes. Are the the classification problems for the p-local and q-local torsion-free abelian groups of rank n comparable with respect to Borel reducibility?*

Theorem 11 (S.T. 2002) *If* $n \geq 3$ *and* $p \neq q$ *are distinct primes, then* $\cong_n^{(p)}$ *and* $\cong_n^{(q)}$ *are incomparable with respect to Borel reducibility.*

Corollary 12 (S.T. 2002) *If* $n \geq 3$ *and* p *is a prime, then* $(\cong_n^{(p)}) <_B (\cong_n)$.

First note that if A, $B \in R^{(p)}(\mathbb{Q}^n)$, then

$$A \cong B \quad \text{iff} \quad \exists g \in GL_n(\mathbb{Q}) \quad g(A) = B.$$

We must show that there doesn't exist a Borel map

$$f : R^{(p)}(\mathbb{Q}^n) \to R^{(q)}(\mathbb{Q}^n)$$

A, B lie in the same $GL_n(\mathbb{Q})$-orbit iff $f(A)$, $f(B)$ lie in the same $GL_n(\mathbb{Q})$-orbit.

First we apply the Kurosh-Malcev localisation.

Definition 13 *For each* $A \in R^{(p)}(\mathbb{Q}^n)$*, let* $\widehat{A} = \mathbb{Z}_p \otimes A$.

We regard each $\widehat{A}$ as a subgroup of $\mathbb{Q}_p^n$ in the usual way. There exist elements v_i, $w_j \in \widehat{A}$ such that

$$\widehat{A} = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p w_j.$$

Definition 14 *For each* $A \in R^{(p)}(\mathbb{Q}^n)$*, let* $V_A = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i$.

Definition 15 *$A \approx B$ iff $A \cap B$ has finite index in both A and B.*

Definition 16 *$A \sim B$ iff there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx B$.*

Theorem 17 *If A, $B \in R^{(p)}(\mathbb{Q}^n)$, then*

(a) $A \approx B$ iff $V_A = V_B$.

(b) $A \sim B$ iff there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(V_A) = V_B$.

Thus the isomorphism relation $\cong_n^{(p)}$ on $R^{(p)}(\mathbb{Q}^n)$ is *virtually* the same as the orbit equivalence relation of $GL_n(\mathbb{Q})$ on the set of vector subspaces of $\mathbb{Q}_p^n$.

Definition 18 *If* $0 \leq k \leq n$*, then* $V^{(k)}(n, \mathbb{Q}_p)$ *denotes the standard Borel space consisting of the* k*-dimensional vector subspaces of* $\mathbb{Q}_p^n$*.*

Theorem 19 *Suppose that* $A \in R^{(p)}(\mathbb{Q}^n)$ *and that* $\dim V_A = n - 1$*. Then for each* $B \in R^{(p)}(\mathbb{Q}^n)$*, we have that* $A \sim B$ *iff* $A \cong B$*.*

Theorem 20 *There exists a Borel map*

$$\sigma : V^{(n-1)}(n, \mathbb{Q}_p) \to R^{(p)}(\mathbb{Q}^n)$$

such that $V_{\sigma(S)} = S$*.*

Suppose that there exists a Borel reduction

$$h : R^{(p)}(\mathbb{Q}^n) \to R^{(q)}(\mathbb{Q}^n).$$

Consider the Borel map

$$f : V^{(n-1)}(n, \mathbb{Q}_p) \to \bigcup_{\ell} V^{(\ell)}(n, \mathbb{Q}_q)$$

defined by

$$S \mapsto \sigma(S) \mapsto (h \circ \sigma)(S) \mapsto V_{h \circ \sigma(S)}.$$

Clearly f is countable-to-one and maps $GL_n(\mathbb{Q})$-orbits to $GL_n(\mathbb{Q})$-orbits.

Consider the ergodic action of $SL_n(\mathbb{Z})$ on

$$(V^{(n-1)}(n, \mathbb{Q}_p), \mu_p)$$

where μ_p is the p-adic probability measure induced by the transitive action of $SL_n(\mathbb{Z}_p)$ on $V^{(n-1)}(n, \mathbb{Q}_p)$.

Wlog there exists a fixed ℓ such that

$$f : V^{(n-1)}(n, \mathbb{Q}_p) \to V^{(\ell)}(n, \mathbb{Q}_q).$$

Clearly f is countable-to-one and maps $SL_n(\mathbb{Z})$-orbits to $GL_n(\mathbb{Q})$-orbits.

A Robust Punchline

Suppose that $p \neq q$ are distinct primes and that $0 < k, \ell < n$. Then

$$(SL_n(\mathbb{Z}), V^{(k)}(n, \mathbb{Q}_p), \mu_p)$$

and

$$(SL_n(\mathbb{Z}), V^{(\ell)}(n, \mathbb{Q}_q), \mu_q)$$

are not isomorphic.

Question: How are we to recognise the prime p in

$$(SL_n(\mathbb{Z}), V^{(k)}(n, \mathbb{Q}_p), \mu_p)?$$

Let Δ be a normal subgroup of finite index in $SL_n(\mathbb{Z})$ and let $cl(\Delta)$ be the closure of Δ in $SL_n(\mathbb{Z}_p)$. Then

$$[SL_n(\mathbb{Z}_p) : cl(\Delta)] = bp^r$$

for some $r \geq 0$ and some divisor b of $|SL_n(\mathbb{F}_p)|$.

The ergodic components for the action of Δ on $V^{(k)}(n, \mathbb{Q}_p)$ are precisely the orbits of $cl(\Delta)$ on $V^{(k)}(n, \mathbb{Q}_p)$ and $SL_n(\mathbb{Z}_p)$ acts transitively on the set of $cl(\Delta)$-orbits.

Hence the number of ergodic components

$$e_p(\Delta) = cp^s$$

for some $s \geq 0$ and some divisor c of $|SL_n(\mathbb{F}_p)|$.

This argument shows that

$$(SL_n(\mathbb{Z}), V^{(k)}(n, \mathbb{Q}_p), \mu_p)$$

and

$$(SL_n(\mathbb{Z}), V^{(\ell)}(n, \mathbb{Q}_q), \mu_q)$$

are not virtually isomorphic.

More generally,

$$(SL_n(\mathbb{Z}), V^{(\ell)}(n, \mathbb{Q}_q), \mu_q)$$

is not a virtual factor of any finite ergodic extension of

$$(SL_n(\mathbb{Z}), V^{(k)}(n, \mathbb{Q}_p), \mu_p).$$

Recall that

$$f : X = V^{(n-1)}(n, \mathbb{Q}_p) \to Y = V^{(\ell)}(n, \mathbb{Q}_q)$$

is countable-to-one and maps $SL_n(\mathbb{Z})$-orbits to $GL_n(\mathbb{Q})$-orbits.

Wlog there are finitely primes such that f maps $SL_n(\mathbb{Z})$-orbits to $SL_n(\mathbb{Z}[1/q_1, \dots, 1/q_t])$-orbits.

Notation:

$$\Lambda = SL_n(\mathbb{Z}[1/q_1, \dots, 1/q_t])$$

$$H = SL_n(\mathbb{R}) \times SL_n(\mathbb{Q}_{q_1}) \times \cdots \times SL_n(\mathbb{Q}_{q_t})$$

We regard $K = SL_n(\mathbb{Z}_{q_1}) \times \cdots \times SL_n(\mathbb{Z}_{q_t})$ and $SL_n(\mathbb{R})$ as (commuting) subgroups of H.

Slightly oversimplifying ...
Zimmer superrigidity yields a Borel embedding of permutation groups

$$(SL_n(\mathbb{Z}), X, \mu_p) \xrightarrow{\varphi, \widetilde{f}} (SL_n(\mathbb{R}), Y \times (K \backslash H/\Lambda), \widetilde{f}_* \mu_p)$$

We must study the distribution of $\widetilde{f}(X)$ within $Y \times (K \backslash H/\Lambda)$.

There are only countably many $SL_n(\mathbb{R})$-orbits on $K \backslash H/\Lambda$. So wlog there exists a single $SL_n(\mathbb{R})$-orbit Ω on $K \backslash H/\Lambda$ such that

$$\widetilde{f}(X) \subseteq Y \times \Omega.$$

We can define a $\varphi(SL_n(\mathbb{Z}))$-invariant ergodic probability measure ω on Ω by

$$\omega(A) = \mu_p(\{x \in X \mid \tilde{f}(x) \in Y \times A\}).$$

Furthermore, (Ω, ω) is a factor of (X, μ_p).

Ratner's Theorem implies that ω is either the Haar measure or else has finite support. The dynamical properties of (X, μ_p) rule out the first possibility. So there exists a finite $\varphi(SL_n(\mathbb{Z}))$-invariant subset Ω_0 such that

$$\tilde{f}(X) \subseteq Y \times \Omega_0.$$

This implies that

$$(SL_n(\mathbb{Z}), V^{(k)}(n, \mathbb{Q}_p), \mu_p)$$

and

$$(SL_n(\mathbb{Z}), V^{(\ell)}(n, \mathbb{Q}_q), \mu_q)$$

are virtually isomorphic, which is a contradiction.