Planar Algebra of the Subgroup-Subfactor

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Overview

- Given a pair of finite groups $H \leq G$, an outer action α of G on the hyperfinite II_1-factor R gives rise to the (hyperfinite) subgroup-subfactor

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- Further, given a finite bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ with a spin function $\mathcal{U}^+ \sqcup \mathcal{U}^- \xrightarrow{\mu} (0, \infty)$, Jones associated a planar algebra $P(\Gamma)$.

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- $[G : H] = n$. The obvious G action on G/H yields an action of G on the bipartite graph $\star_n$ (with $|\mathcal{U}^+| = n$, $|\mathcal{U}^-| = 1$, and the spin function whose entrywise squares gives the 'Perron-Frobenius eigenvector'), the G invariant planar subalgebra of $P(\star_n)$ is isomorphic to the planar algebra $P^{R \rtimes H \subset R \rtimes G}$.

  (This last result is mentioned, although without any indication of proof, in the 'preprepreprint' [Jon03], which I came to know of only after this work was done.)

Outline

- Concrete model for the basic construction tower of $R \rtimes H \subset R \rtimes G$.
- "Orbit bases" for relative commutants in terms of the model tower.
- Planar Algebra of a bipartite Graph and G-action.
- Planar Algebra of the Subgroup-Subfactor:

$$P^{R\rtimes H \subset R \rtimes G} \cong P(\star_n)^G.$$

- Planar Algebra of the fixed subfactor:

$$P^{R^G \subset R^H} \cong P(\overline{\star_n})^G.$$

Proposition. $N \subset M \subset^{e_1} M_1$ be the basic construction for a subfactor $N \subset M$ with $[M : N] < \infty$. For any finite index set Λ,

$$M_\Lambda(N) \subset M_\Lambda(M) \subset M_\Lambda(M_1)$$

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Lemma. Consider $N \subset M$ with $n = [M : N] \in \mathbb{N}$, and an orthonormal basis $\{\lambda_i : i \in I\}$ (i.e., $E_n(\lambda_i \lambda_j^*) = \delta_j^i, \forall i, j \in I$), $I := \{1, \ldots, n\}$. Then

$$\begin{aligned}\left(N \subset M \overset{\theta}{\hookrightarrow} M_I(N) \subset M_I(M)\right) &\cong (N \subset M \subset M_1 \subset M_2), \text{ where} \\ \theta_{i,j}(x) &:= E_N(\lambda_i x \lambda_j^*), \forall\, x \in M,\ i, j \in I.\end{aligned}$$

Fix an outer action α of a finite group G on the hyperfinite II_1-factor R.
$H \leq G$; $G = \sqcup_{i=1}^{n} Hg_i$, with $g_1 = e$. We have

$$R \rtimes G = \left\{ \sum_g x_g u_g : x_g \in R \right\} \subset \mathcal{L}(L^2(R)), \text{ where } u_g x = \alpha_g(x) u_g.$$

$\{u_{g_i} : 1 \leq i \leq n\}$ is an orthonormal basis for $N := R \rtimes_{\alpha/H} H \subset R \rtimes_\alpha G =: M$.

$$(N \subset M \subset M_1 \subset M_2) \cong (N \subset M \overset{\theta}{\hookrightarrow} M_I(N) \subset M_I(M));$$

and

$$\begin{array}{c} M_{2k-1} \subset M_{2k} \subset M_{2k+1} \subset M_{2k+2} \\ \cong \\ M_{I^k}(N) \subset M_{I^k}(M) \overset{\Theta_{k+1}}{\hookrightarrow} M_{I^{k+1}}(N) \subset M_{I^{k+1}}(M), \end{array}$$

$\forall\, k \geq 0$, where $\Theta_{k+1} := M_I(\Theta_k)$ with $\Theta_1 := \theta$.

Theorem

$$N \subset M \overset{\Theta_1}{\hookrightarrow} M_I(N) \subset M_I(M) \overset{\Theta_2}{\hookrightarrow} \cdots \subset M_{I^k}(M) \overset{\Theta_{k+1}}{\hookrightarrow} M_{I^{k+1}}(N) \subset \cdots$$

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is a model for the basic construction tower of the subgroup-subfactor $N := R \rtimes H \subset R \rtimes G =: M$.

M sits in $M_{2k-1} \cong M_{I^k}(N)$ by the map $\Theta_k \circ \cdots \circ \Theta_1 = \theta^{(k)}$ given by

$$\theta^{(k)}_{\underline{i}, \underline{j}}(x) \quad = \quad \theta_{i_1, j_1}(\theta_{i_2, j_2}(\cdots \theta_{i_k, j_k}(x) \cdots))$$

It turns out ([JS97]) that there is a G-action[1] on $I^k, k \geq 1$, such that the set $Y_k := \{(\underline{i}, \underline{j}) \in I^k \times I^k : H \sqcap g_{\underline{i}} = H \sqcap g_{\underline{j}}\}$ - where $\sqcap g_{\underline{i}} := g_{i_1} g_{i_2} \cdots g_{i_k}$ - is invariant under the diagonal action of G.

Further, $N' \cap M_{2k-1}$ has a basis indexed by $H \backslash Y_k$, the space of H-orbits of Y_k; write $[\underline{i}, \underline{j}]^{od}$ for the basis vector of $N' \cap M_{2k-1}$ corresponding to the H-orbit of $(\underline{i}, \underline{j})$.

Similarly, $N' \cap M_{2k}$ has a basis indexed by $H \backslash (I^k \times I^k)$, the space of H-orbits of $I^k \times I^k$; write $[\underline{i}, \underline{j}]^{ev}$ for the basis vector of $N' \cap M_{2k}$ corresponding to the H-orbit of $(\underline{i}, \underline{j})$.

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$$g \cdot \underline{j} = \underline{i} \Longleftrightarrow H g_{j_s} g_{j_{s+1}} \cdots g_{j_k} g^{-1} = H g_{i_s} g_{i_{s+1}} \cdots g_{i_k}, \ \forall 1 \leq s \leq k.$$

- $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$: connected, bipartite, finite (multi-) graph.
  Spin function: $\mu : \mathcal{U}^+ \sqcup \mathcal{U}^- \to (0, \infty)$.

- Jones [Jon00] gave a planar algebra structure on
  $P(\Gamma) := \{P_k(\Gamma) \ : \ k \in Col \ := \{0_\pm, 1, 2, \ldots\}\}$, where
  $P_{\pm 0}(\Gamma) := \mathbb{C}[\mathcal{U}^\pm]$ and
  $P_k(\Gamma) := \mathbb{C}[\text{loops of length } 2k \text{ on } \Gamma \text{ based at vertices in } \mathcal{U}^+]$ for $k \geq 1$.

- For the reversed graph $\overline{\Gamma} := (\mathcal{U}^-, \mathcal{U}^+, \mathcal{E})$, we have $P(\overline{\Gamma}) \cong {}^-P(\Gamma)$.[2]

[2] Recall ([KS04]) that each planar algebra P admits a dual planar algebra ${}^-P$ in such a way that ${}^-P^{N \subset M} \cong P^{M \subset M_1}$.

A finite group G acts on (Γ, μ), if

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Such an action induces a G-action on the planar algebra $P(\Gamma)$, i.e., G commutes with the tangle actions on $P(\Gamma)$, and we have the planar subalgebra

$$P(\Gamma)^G := \{P(\Gamma)^G_k; k \in \; Col\}$$

of $P(\Gamma)$, where $P(\Gamma)^G_k := \{x \in P_k(\Gamma) : g \cdot x = x, \forall g \in G\}$.

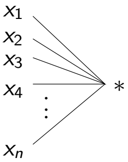
The above G-action induces a G-action on the dual $^-P(\Gamma) \cong \; P^{(}\overline{\Gamma})$, and

$$P(\overline{\Gamma})^G \; \cong \; ^-(P(\Gamma)^G).$$

The Bipartite Graph $\star_n$

$G = \sqcup_{i=1}^{n} Hg_i$, with $g_1 = 1$. We write X for $H\backslash G$ and $x_i := Hg_i, 1 \leq i \leq n$. We have

$\Gamma = \star_n :=$



and $\mu(x_i) = 1,\ \forall\ i$ and $\mu(*) = n^{1/4}$.

The G action on $H\backslash G$ yields the G-action on the bipartite graph $\star_n$:

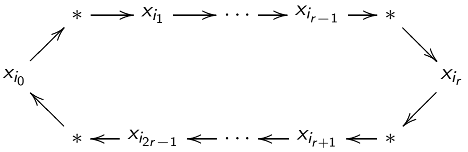
$$g \cdot x_i = x_j \text{ iff } Hg_i g^{-1} = Hg_j; \text{ and } g \cdot * = *, \forall g \in G.$$

This induces a G action on the planar algebra $P(\star_n)$, and we get a connected, irreducible planar algebra $P(\star_n)^G$ with positive modulus $\sqrt{n}$.

If $k > 0$ is even, say $k = 2r$, we simply write

$$\begin{pmatrix} & x_{i_1}, & \cdots & , x_{i_{r-1}} & \\ x_{i_0} & & & & x_{i_r} \\ & x_{i_{2r-1}}, & \cdots & , x_{i_{r+1}} & \end{pmatrix}$$

for the $2k$-loop



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By definition,

$$\left\{ \left(\begin{matrix} & x_{i_1}, & \cdots & , x_{i_{r-1}} & \\ x_{i_0} & & & & x_{i_r} \\ & x_{i_{2r-1}}, & \cdots & , x_{i_{r+1}} & \end{matrix} \right) : (i_0, i_1, \ldots, i_{2r-1}) \in I^{2r} \right\}$$

forms a basis for $P_{2r}(\star_n)$, which is seen to be mapped into itself by the G-action. Hence, the distinct elements of the set

$$\left\{ \left[\begin{matrix} & x_{i_1}, & \cdots & , x_{i_{r-1}} & \\ x_{i_0} & & & & x_{i_r} \\ & x_{i_{2r-1}}, & \cdots & , x_{i_{r+1}} & \end{matrix} \right] := \sum_{g \in G} g \left(\begin{matrix} & x_{i_1}, & \cdots & , x_{i_{r-1}} & \\ x_{i_0} & & & & x_{i_r} \\ & x_{i_{2r-1}}, & \cdots & , x_{i_{r+1}} & \end{matrix} \right) : \underline{i} \in I^{2r} \right\}$$

form a basis for $P_{2r}(\star_n)^G$.

A similar analysis is seen to hold for odd k.

Finally, $\{\sum_{i \in I} x_i\}$ and $\{*\}$ form bases for $P_{0_+}(\star_n)^G$ and $P_{0_-}(\star_n)^G$, respectively.

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We need to define maps

$$\varphi_k : P_k^{R\rtimes H \subset R\rtimes G} = N' \cap M_{k-1} \to P_k(\star_n)^G, \ \forall \ k \in \ Col.$$

Set

$$\varphi_{0_\pm} = \varphi_1 = id_{\mathbb{C}}; \text{ and}$$

define

$$\begin{aligned} \varphi_{2r}([\underline{i}, \underline{j}]^{od}) &= \left[x_1 \begin{matrix} x_{p_r}, \cdots, x_{p_2} \\ x_{q_r}, \cdots, x_{q_2} \end{matrix} x_{q_1} \right], \text{ and} \\ \varphi_{2r+1}([\underline{i}, \underline{j}]^{ev}) &= \left[x_1 \begin{matrix} x_{p_r}, \cdots, x_{p_2}, x_{p_1} \\ x_{q_r}, \cdots, x_{q_2}, x_{q_1} \end{matrix} \right], \end{aligned}$$

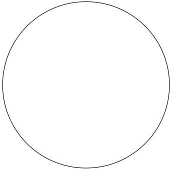
where $x_{p_l} = Hg_{i_l} g_{i_{l+1}} \cdots g_{i_r}$ and $x_{q_l} = Hg_{j_l} g_{j_{l+1}} \cdots g_{j_r}$ for $1 \leq l \leq r$.

In order to prove that $\varphi = \{\varphi_k : k \in Col\}$ is a planar algebra isomorphism, we need to verify the maps φ_k are equivariant with respect to the tangle actions. Fortunately, this needs to be done only for a 'generating class of tangles', as in:

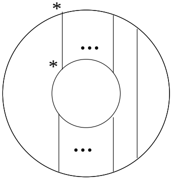
Theorem: [KS04] Let $\mathcal{T}$ be a collection of planar tangles containing

$$\{1^{0\pm}\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\} \cup \{\mathcal{E}^{k+1}, (E')^k_k : k \geq 1\},$$

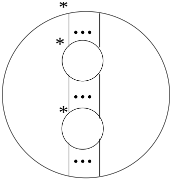
and suppose $\mathcal{T}$ is closed under composition, whenever it makes sense. Then $\mathcal{T}$ contains all planar tangles.



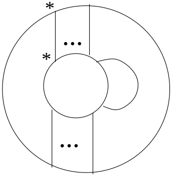
1^{0_+}



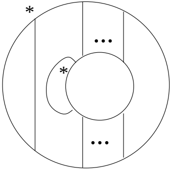
I_k^{k+1}



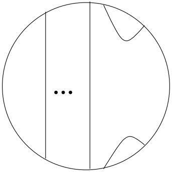
M_k



E_{k+1}^k



$(E')_k^k$



$\mathcal{E}^{k+1}$

Immediate Observations

Repeated applications of "Not-Burnside's Lemma" gives:

Corollary: *dim* $P_k^{R\rtimes H\subset R\rtimes G} = \frac{1}{|G|}\sum_{C\in\mathcal{C}_G}|C|\left(\frac{|C\cap H||G|}{|C||H|}\right)^k$, $k\geq 1$, where $\mathcal{C}_G$ is the set of conjugacy classes of G.

Our result yields unexpected universal 'upper and lower bounds' for the planar algebra of any index n subgroup-subfactor.

Corollary: Given any pair of finite groups $H\subset G$ with index n,

$$P^{R\rtimes S_{n-1}\subset R\rtimes S_n}\cong P(\star_n)^{S_n}\subset P^{R\rtimes H\subset R\rtimes G}\subset P(\star_n).$$

With $N := R \rtimes H \subset R \rtimes G =: M$, we have $(M \subset M_1) \cong (R^G \subset R^H)$.

Recall that $P^{M \subset M_1} \cong {}^{-}P^{N \subset M}$ and $P(\overline{\Gamma})^G \cong {}^{-}(P(\Gamma)^G)$. Thus, we have:

Corollary: $P^{R^G \subset R^H} \cong P(\overline{\star_n})^G$.

Corollary: If $[G : H] = n$ then

$$P^{R^{S_n} \subset R^{S_{n-1}}} \cong P(\overline{\star_n})^{S_n} \subset P^{R^G \subset R^H} \subset P(\overline{\star_n}).$$

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