Multitaper Spectrum Estimates and Some Applications

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RSC, Killam Foundation, CRC, Queen's, NSERC,
(the old) Bell Labs (Murray Hill, NJ), &
Maja–Lisa Thomson

Estimating Power Spectrum:

Stationary process, $x(t)$, $\quad t = -1, 0, 1, \cdots$
Zero mean, $\mathbf{E}\{x(t)\} = 0$, finite variance $\mathbf{E}\{x^2(t)\} = \sigma^2 < \infty$

Spectral Representation (Cramér, Doob):

$$x(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi ft} dX(f)$$

Power Spectral Density, or Spectrum $S(f)$

$$S(f)df = \mathbf{E}\{|\, dX(f)\,|^2\}$$

(= Fourier Transform of the autocovariance)

Problem: <u>Estimate</u> $S(f)$ from a finite data sample

Some Personal History — I

(1961) Started at Acadia. Tried to read *"Prolate Spheroidal Wave Functions (PSWF), Fourier Analysis and Uncertainty"*

(1964) "... there is a lot of work being done on time–series, but I have a feeling it's on the wrong track." — K. D. C. Haley

(1965) Started work at Bell Labs, Murray Hill, NJ.

Tukey & Cooley published the FFT

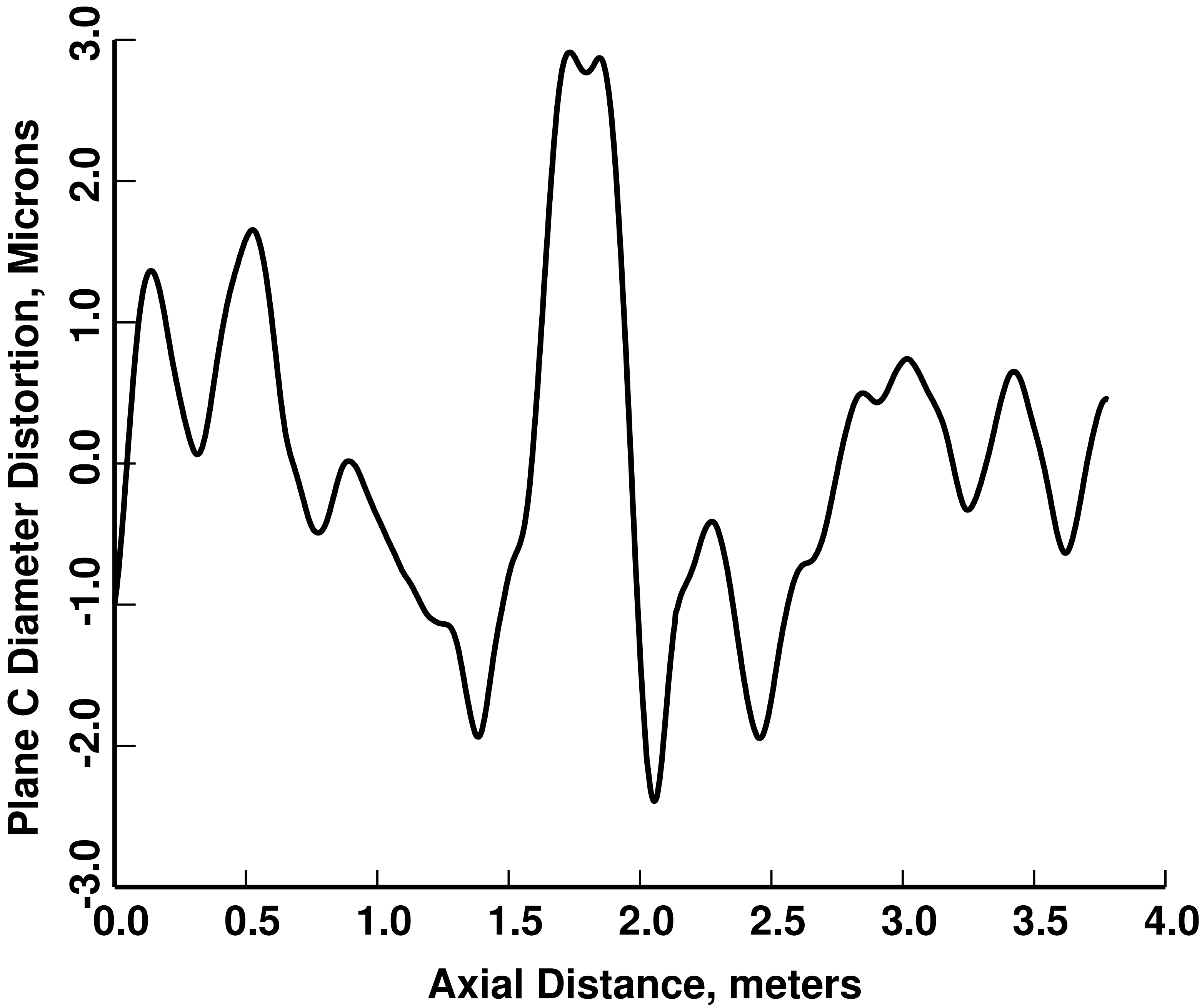
(1966) "Millimeter Waveguide Project": Spectrum estimation of geometric distortions (mode–conversion loss).

(1967) Computing spectra: mean–value function, prewhitening, data taper, FFT, correct for prewhitening.

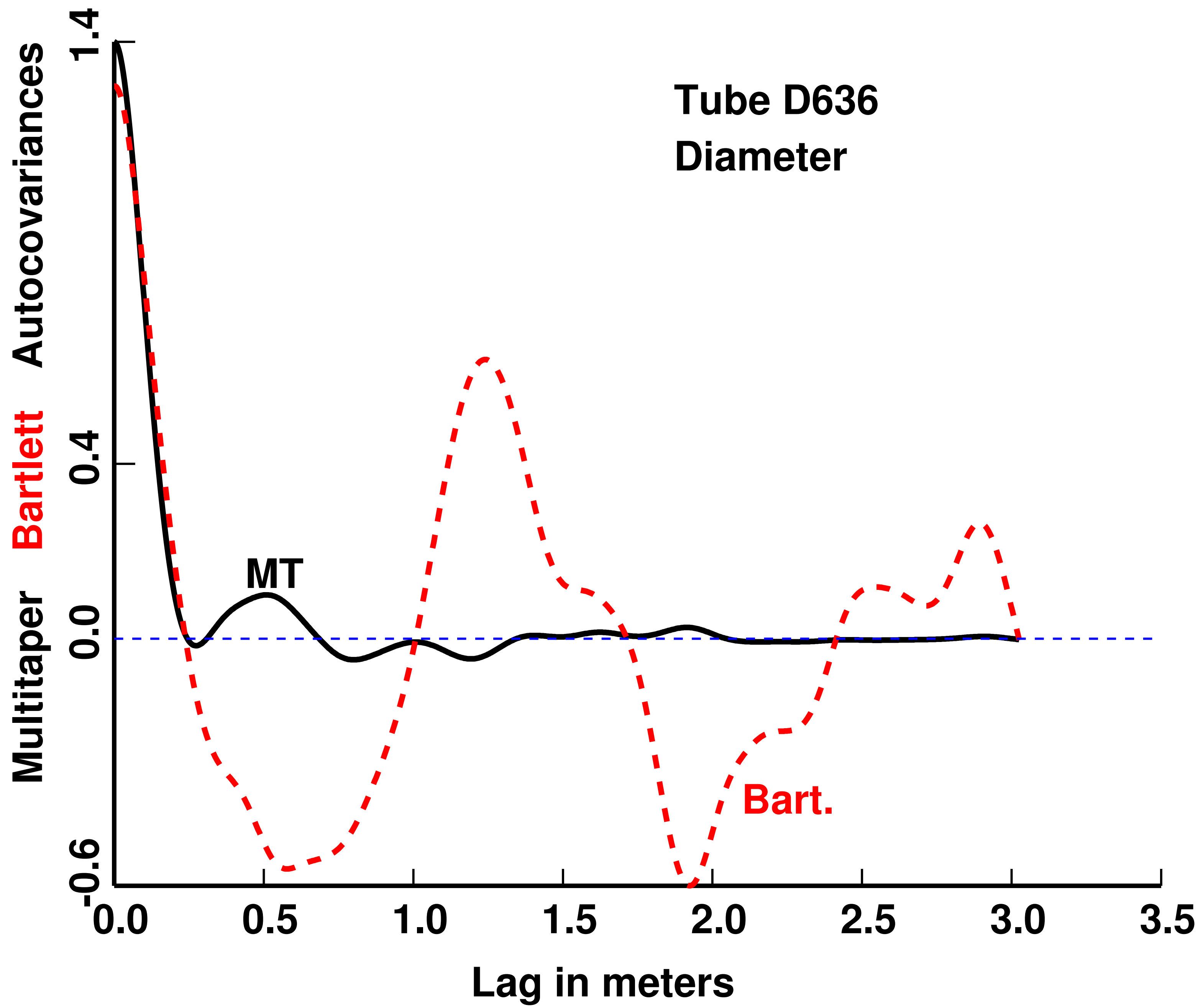
(c1968) PSWF data taper with FFT → PhD (1971).

(c1972) Tried a multitaper with 2 prolate windows. It "split lines", so dropped it (temporarily).

WT4 Waveguide Diameter vs Distance



Waveguide Autocovariances (Multitaper & Bartlett)



Some Personal History — II

(c1974) "I can find at least 100 different ways to estimate the spectrum in the literature. What I want to know from you is which is the right one, and why. And, if you are wrong, we are talking about a measurable fraction of the US GNP"

— D. A. Alsberg

1) Using estimated spectrum for quality control

2) Expensive, $\approx$ cost of a Volkswagen turned to scrap

3) What if the system didn't work?

Windowed Estimates, Tukey (1959, 1966)

Using data "taper" $D(n)$

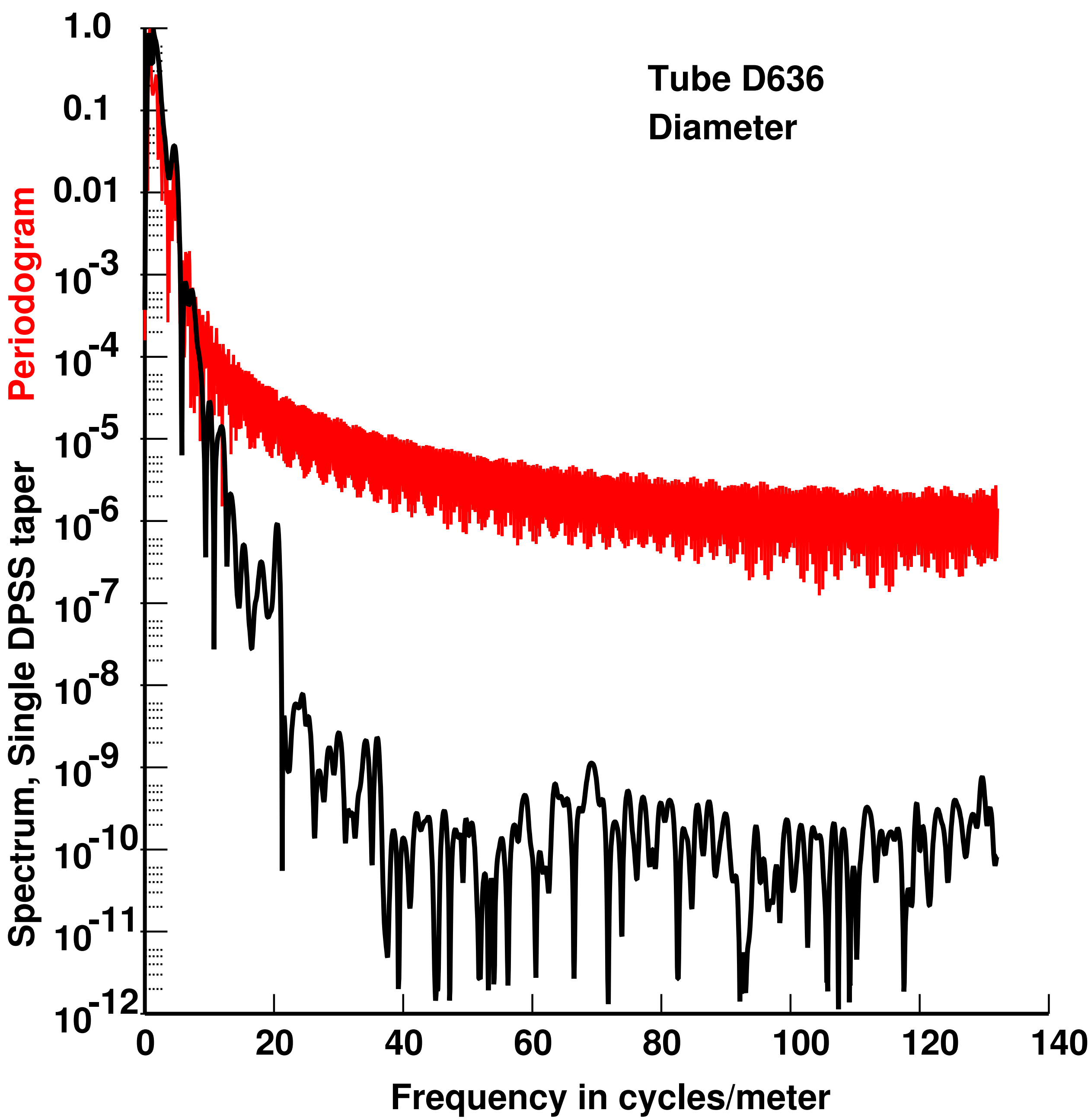
$$\widehat{S}_D(f) = \left| \sum_{n=0}^{N-1} x(n) D(n) e^{-i2\pi n f} \right|^2$$

Good choice of D gave obviously better estimates of $S(f)$

$$\mathbf{E}\{\widehat{S}_D(f)\} = S(f) \bigstar \left| \sum_{n=0}^{N-1} D(n) e^{-i2\pi n f} \right|^2$$

Many "Optimum" $D(n)$'s

Spectrum Estimates for WT4 Waveguide Diameter Data



Periodogram:
$D(n) =$ constant

Multitaper:
$D(n) =$ PSWF

Peaks: machinery
— Noisy Estimate

Windowed Estimates, Tukey (1959, 1966)

A problem $\cdots$

Only theory: John Tukey said it's a good idea!

"If one were not blinded by the mathematical elegance of the conventional approach, making unfounded assumptions as to the values of unmeasured data and changing the data values that one knows would be totally unacceptable"

J. P. Burg *Thesis* (Stanford, 1975)

(Plus quite a few similar.)

Some Personal History — III

(1977) **Bell System Technical Journal** papers written, end of waveguide project. Transfer to Cellular Phone project

(1978) First Rome Air Development Center *Spectrum Estimation Workshop* (**Karhunen–Loeve in the frequency domain**)

PSWF Tapers in the Time Domain

I knew it worked, but not why

(1978) D. Slepian, *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty V: The Discrete Case*
Now called "Slepian Sequences and functions"

(1981) **ASSP Workshop on Spectral Estimation**, McMaster

(1982) **Spectrum Estimation and Harmonic Analysis**

Proceedings of the IEEE, Simon Haykin, Guest Editor

Explained why **several** *tapers were needed*

Multitapers — I: Fundamental Integral Equation

Fourier Transform of available data, $\bar{t} = (N-1)/2$

$$y(f) = \sum_{t=0}^{N-1} x(t) e^{-i2\pi f(t-\bar{t})} \tag{1}$$

Spectral Representation

$$x(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi\xi(t-\bar{t})} dX(\xi) \tag{2}$$

Fundamental (Fredholm) Integral Equation

$$y(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin N\pi(f-\xi)}{\sin \pi(f-\xi)} dX(\xi) \tag{3}$$

Least–squares solution in Slepian functions

$$\lambda_k V_k(f) = \int_{-W}^{W} \frac{\sin N\pi(f-\xi)}{\sin \pi(f-\xi)} V_k(\xi) d\xi \tag{4}$$

Multitaper Solution

Center frequency f_0, solve on $(f_0 - W, f_0 + W)$

$$y_k(f) = \frac{1}{\lambda_k} \int_{-W}^{W} y(f_0 - \xi)\, V_k(\xi) d\xi \tag{5}$$

$\cdots$ some magic $\cdots$

$$= \sum_{t=0}^{N-1} x(t)\, v_t^{(k)} e^{-i2\pi f t} \tag{6}$$

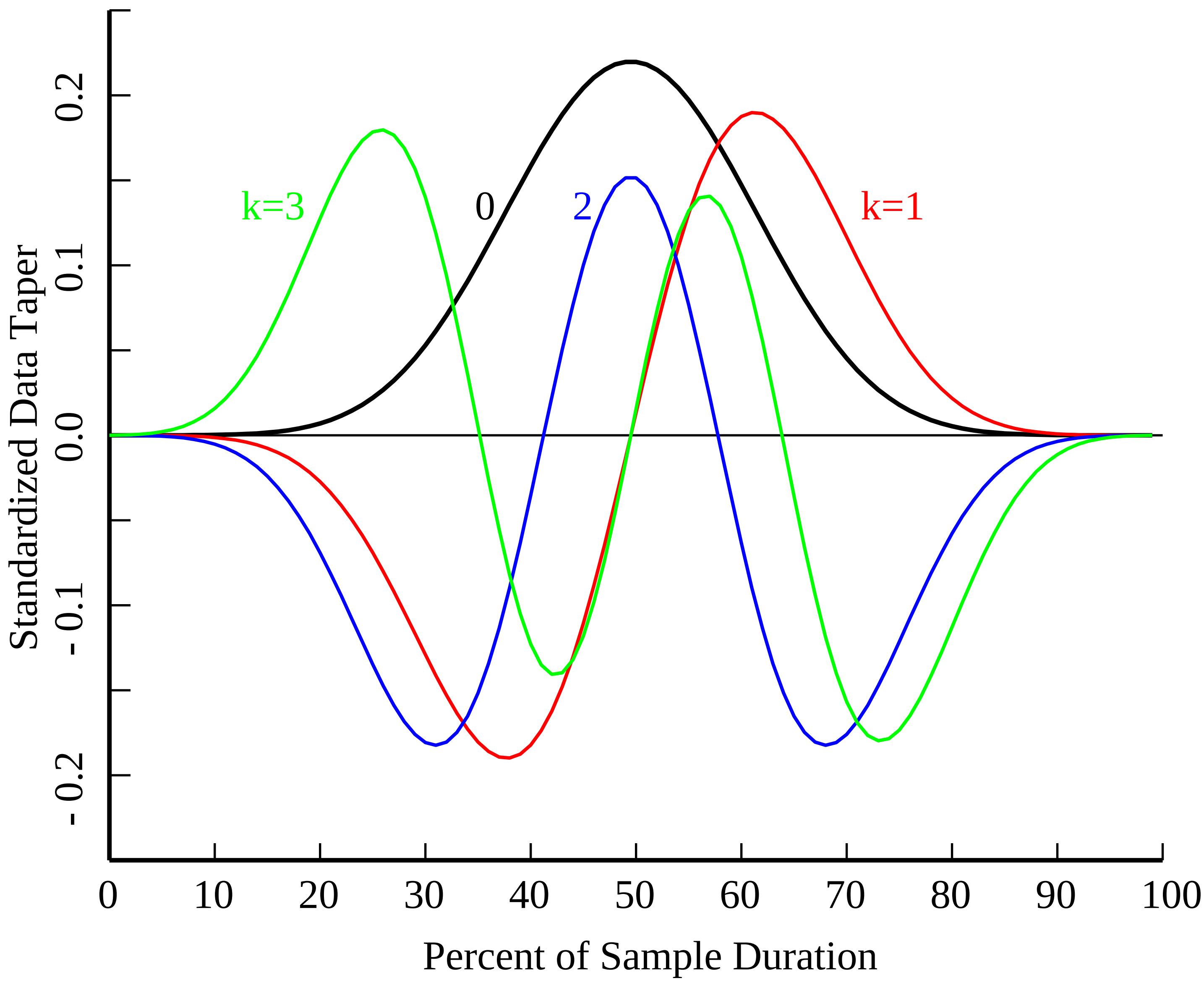
$\Rightarrow$ Fourier Transform of $\{$ data $\times$ taper $\}$

$\Rightarrow$ Orthonormal expansion of $\{$ data $\times$ $e^{-i2\pi f t}$ $\}$

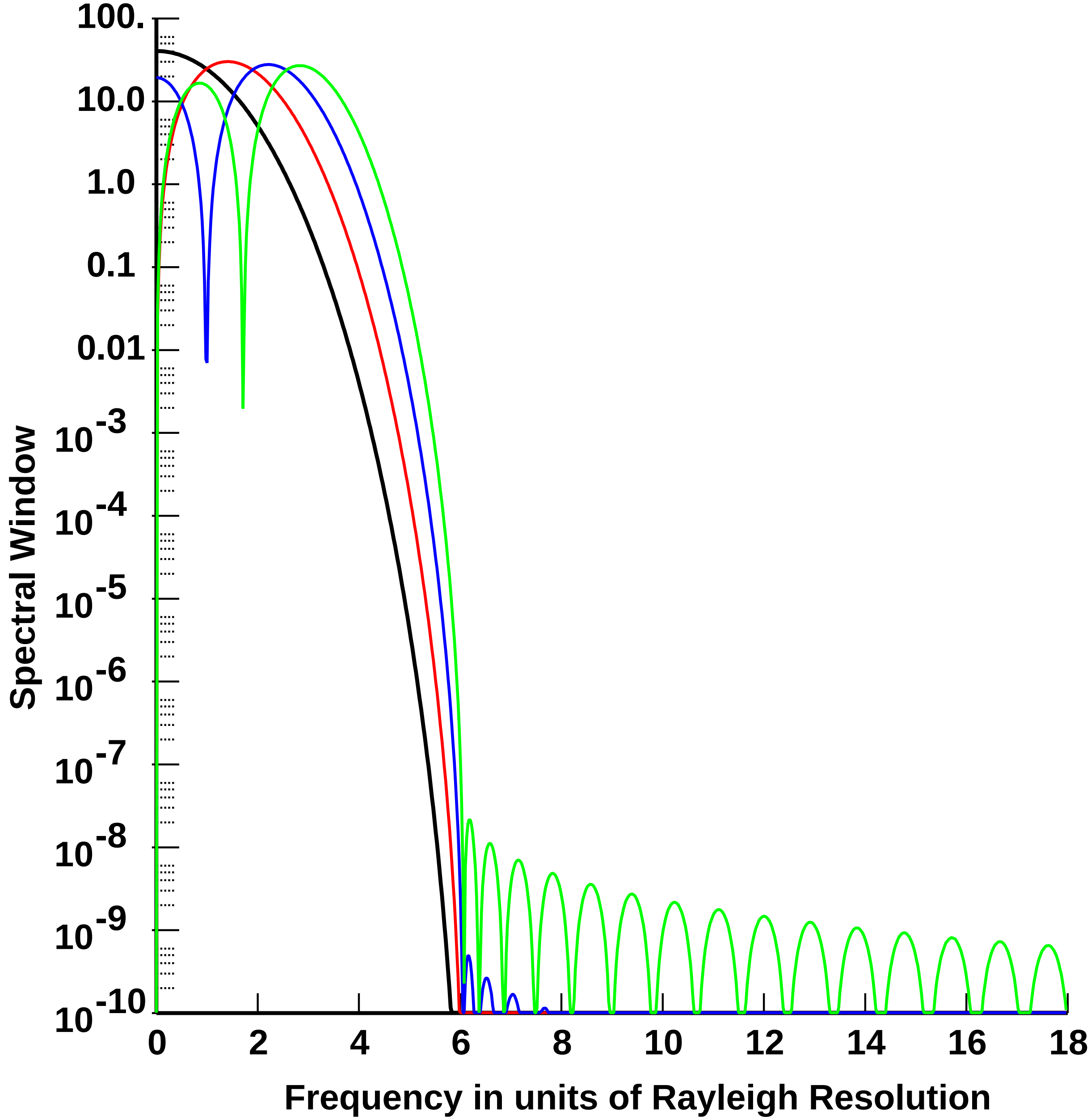
VERY IMPORTANT: $k = 0, 1, \cdots K \approx 2NW$

First term, $y_0(f) \approx$ Tukey's direct estimate (1977 BSTJ)

Slepian Sequences = Data Tapers



Spectral Windows, $|V_k(f)|^2$



Multitaper Spectrum Estimates

Slepian Sequences *define* the dimensionality of the time–frequency region $(-W, W) \times [0, N-1]$.

$$\widehat{dX}(f+\xi) \sim \sum_{k=0}^{K-1} y_k(f) V_k(\xi)\, d\xi \tag{7}$$

Simplest Estimate, Average Power in $(f - W, f + W)$

$$\hat{S}(f) = \frac{1}{N}\frac{1}{2W}\int_{-W}^{W} \left|\widehat{dX}(f+\xi)\right|^2 d\xi \tag{8}$$

$$= \frac{1}{2NW}\sum_{k=0}^{K-1} \lambda_k \left|y_k(f)\right|^2 \approx \frac{1}{K}\sum_{k=0}^{K-1} \left|y_k(f)\right|^2 \tag{9}$$

— A consistent estimate

Where are we?

1) Start: *Finite* sample from a stationary process

2) Take its Fourier transform, invoke spectral representation

3) Get Fredholm integral equation: no unique solution

4) Bash on regardless: do a *local* least–squares solution
   Eigenfunctions are Slepians

5) Eigencoefficients = Fourier transform $\{$ Data $\times$ taper $\}$

6) Split eigencoefficient into *local info* and *broad–band bias*
   Broad–band bias comes from frequencies <u>outside</u>
   $(f - W, f + W)$ — bound and ignore.
   Local: Lots of tools — F–test, Quadratic–Inverse, MSC $\cdots$

Other Side of Fourier Transform — Autocorrelations

Basic Definition: $R_x(\tau) = \mathbf{E}\{x(t+\tau)x^*(t)\} = R(\tau\,;\,\{x\})$

Implies:

1) Quadratic scaling

2) Non-negativity

3) Modulation covariance

$$R(\tau\,;\{e^{i\omega t}x(t)\}) = e^{i\omega\tau}R(\tau\,;\,\{x\})$$

$\Rightarrow$ Need Multiple Taper Estimate (McWhorter & Scharf 1998)

$$\widehat{R}_{mt}(\tau) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{S}_{mt}(f)\, e^{i2\pi f\tau}\, df \tag{10}$$

$$= \frac{1}{K}\sum_{k=0}^{K-1}\sum_{n=\tau}^{N-1} x(n)\, v_n^{(k)}\, x(n-\tau)\, v_{n-\tau}^{(k)} \tag{11}$$

$=$ Average of autocorrelations of $\{\, x(n)\; v_n^{(k)}\,\}$

Other Side of Fourier Transform — Autocorrelations — II

$$\widehat{S}(f) = S(f)\,\frac{\widehat{S}(f)}{S(f)} = S(f) \times G(f) = S(f) \times (1 + \delta G(f))$$

"an observed autocorrelation always exhibits less damping than the theoretical" – Bartlett

$$\widehat{S}(f) \Longleftrightarrow \widehat{R}(\tau) = R(\tau) + R(\tau) \bigstar \delta g(\tau)$$

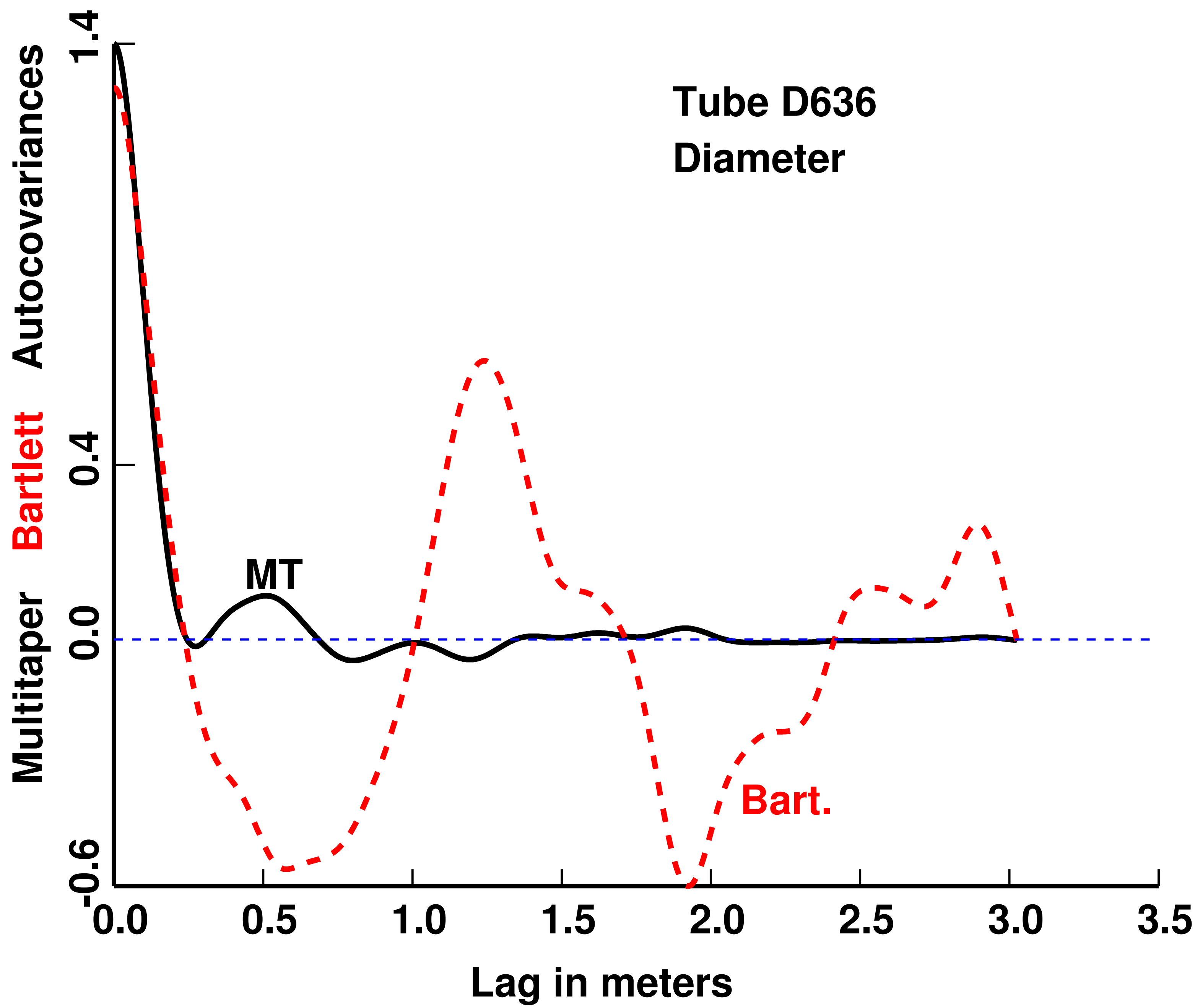
$$\mathbf{E}\{|\delta g(\tau)|^2\} \approx \frac{1}{N}\left(1 - \frac{|\tau|}{N}\right)\left[\frac{\sin 2\pi W\tau}{2\pi W\tau}\right]^2$$

Bartlett autocorrelations

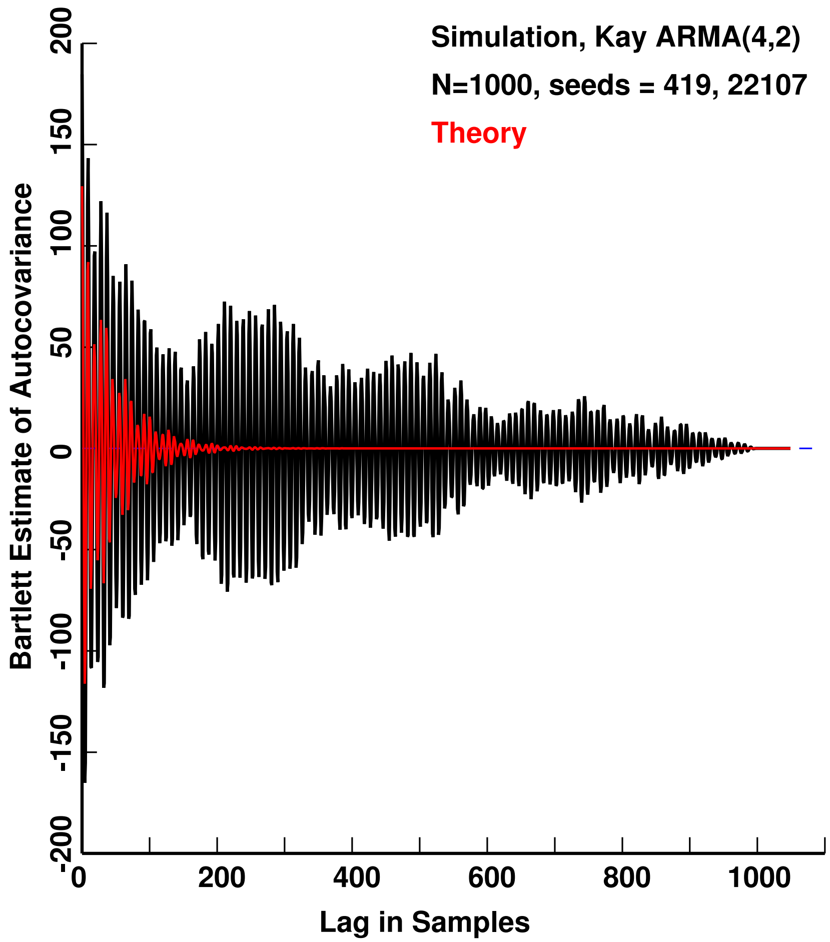
$$\widehat{R}(\tau) = \frac{1}{N}\sum_{n=1}^{N-\tau} x(n)x(n+\tau)$$

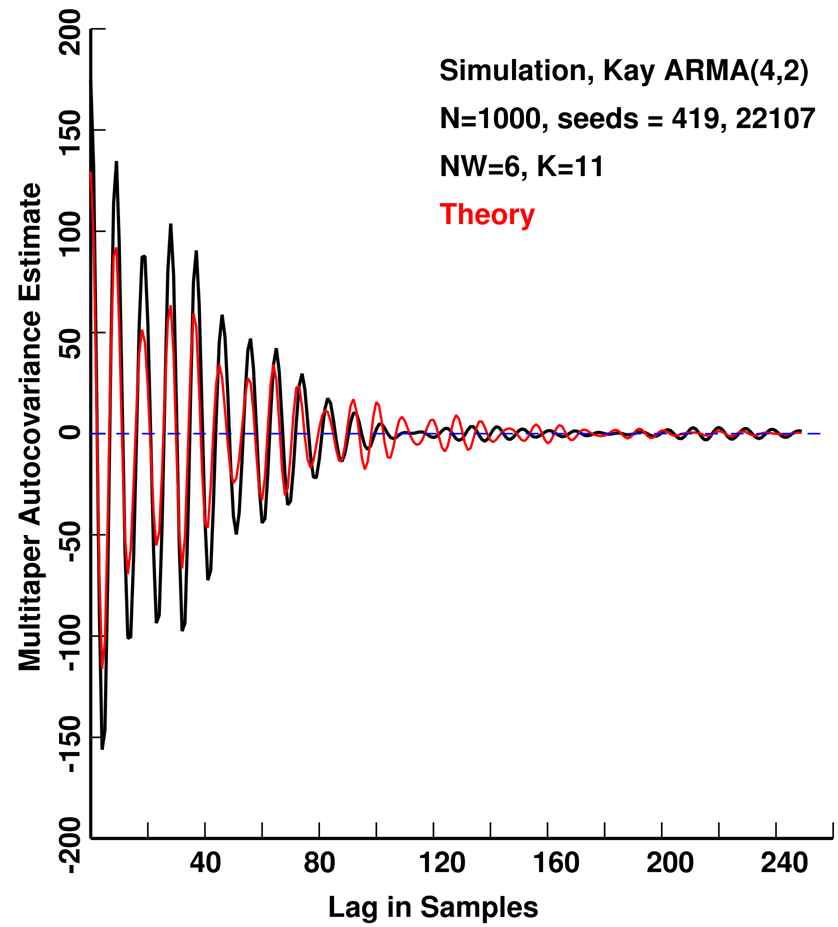
$$= \text{Fourier transform } \{ \text{ Periodogram } \}$$

Waveguide Autocovariances (Multitaper & Bartlett)



Simulation comparing Bartlett & Multitaper Autocovariances





Bartlett Estimate

Multitaper, first 250

Some Personal History — IV

(1983) Green Scholar, *Scripps Institution of Oceanography, San Diego*
Start of switch from **Engineer to Scientist**

(1984) *"Mathematics of Communications Research Department."*
(Shannon's Department at Bell Labs.)

(1985) More Randomness: NSF review by someone at Scripps lead to working with John Imbrie (Brown) on paleoclimate — last 700,000 years (Plus ∼$2 Billion savings for AT&T)

(1989) "Random" Summer Student: Cynthia Kuo

(1990) Climate: **Philosophical Transactions, Royal Society of London, A330, A332**
Coherence between CO_2 and Temperature: **Nature**

(1995) *The Seasons, Global Temperature, and Precession*
"Persistence" — Phil Jones, Comment **Science** (1995)

Part 4: Climate

Harmonic F–test for periodic components

Stationary series plus periodic terms

$$x(t) = x_s(t) + A\cos(2\pi f_o t + \theta)$$

Deterministic mean–value in eigencoefficients

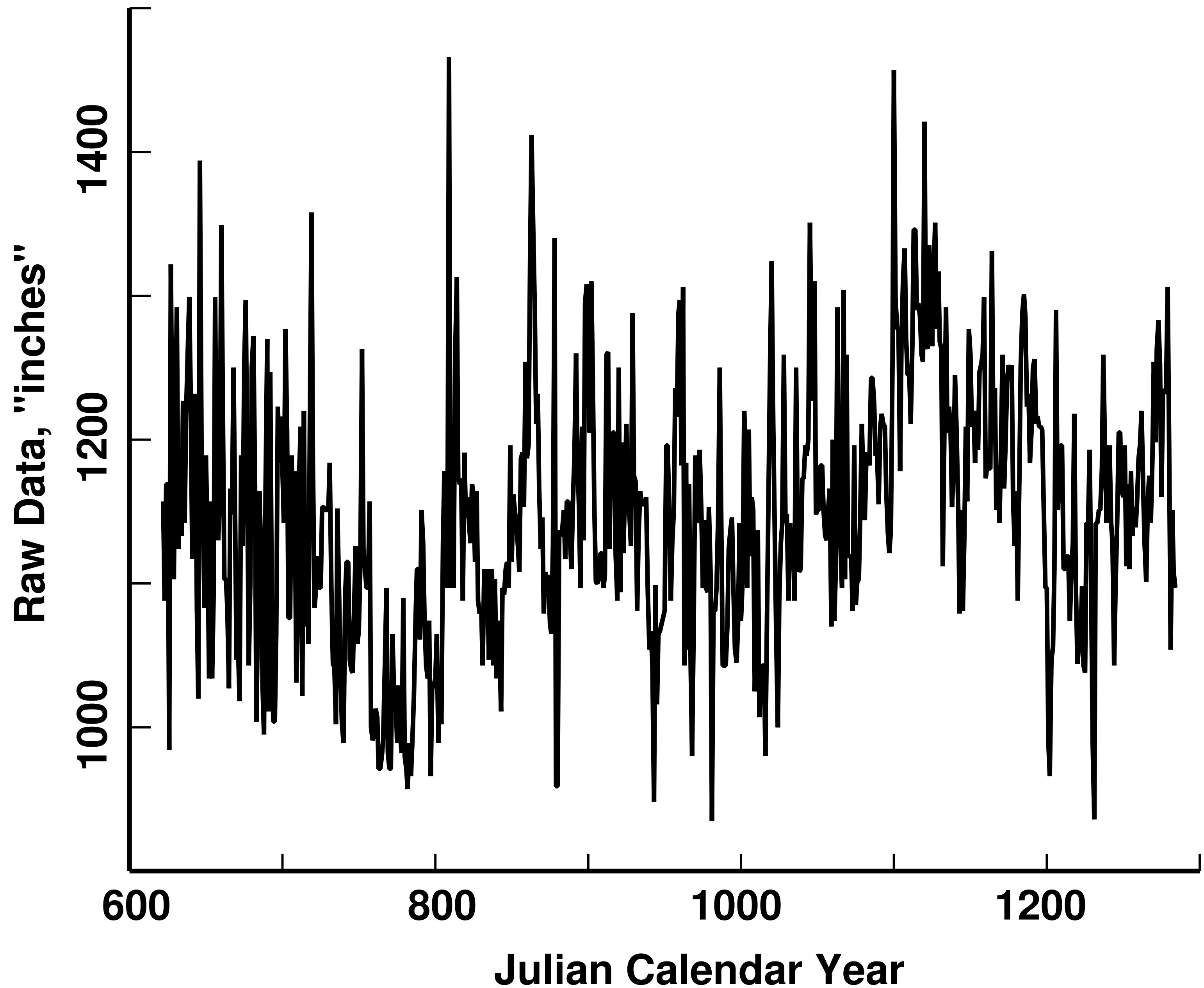
$$y_k(f) = y_{s,k}(f) + \frac{A}{2}e^{i\theta}V_k(f - f_o)$$

Least–squares regression, assume $S(f)$ "locally white"

Ordinary Regression F test

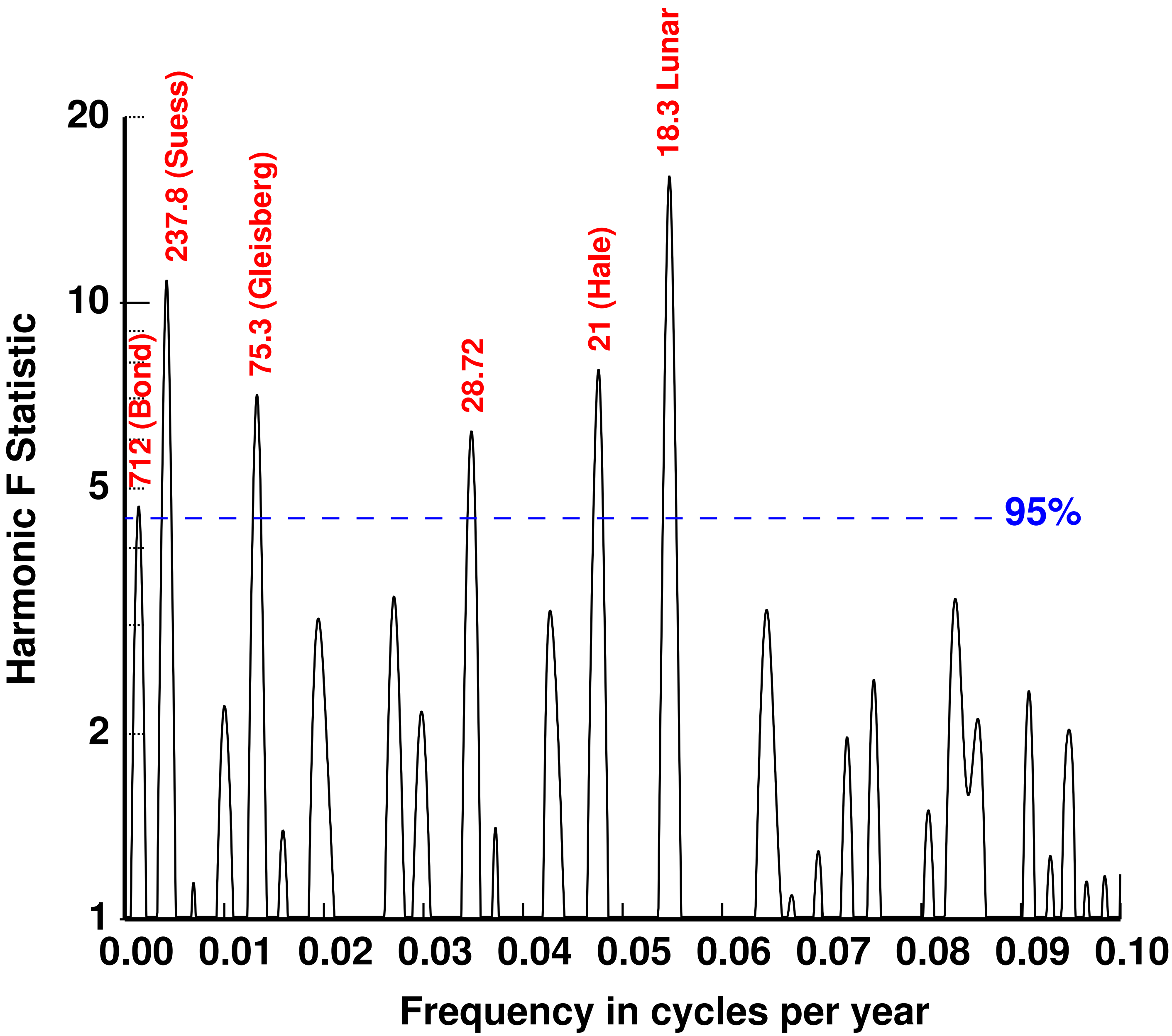
$$F(f) = c\frac{\text{Energy explained by Periodic Term}(f)}{\text{Residual Energy}(f)}$$

Nile River: Yearly Minimum from 622-1284 AD
(Claimed to be Long Memory?)



F-test for Nile River data: very low frequencies

5 out of 6 peaks **named** and **known**



Part 4: Climate — Summary

- *Fluctuations* in $\mathbf{CO_2}$ and temperature are coherent
- Period of annual cycle often not equal to **tropical** year
- **Central England follows general precession constant**
- Solar Irradiance alone does not explain temperature changes (Average and Amplitude of Annual Cycle now *out of phase*)
- **Correct for precession:** $\mathbf{CO_2}$ *and* Solar Irradiance $\sim 10\sigma$
  Solar sensitivity $\approx$ Stefan–Boltzmann
  $\mathbf{CO_2}$: $2.15^{\circ}C/(2 \cdot CO_2)$ — PNAS (1997)
- Many long–term solar effects — **Large**
- Many Short–Period solar oscillations in average and variance

Some Personal History — IV

(1993) Accidental discovery: Solar Modes in Space
(Active Solar Maximum — 1989 blackout)
Communications Satellite "anomalies"
Interplanetary magnetic field, charged particles
Dropped calls on Cell Phones

(1995) DJT, Maclennan, Lanzerotti, *Propagation of solar oscillations through the interplanetary medium*, **Nature**

(1996) Vigorously attacked
— Everyone "**knows**" interplanetary space is "**turbulent**"!

(2009) Ghosh, DJT, Matthaeus, Lanzerotti, *Coexistence of turbulence and discrete modes*, **JGR**

(2007) DJT, Lanzerotti, Vernon, Lessard, Smith *Solar Modal Structure of the Engineering Environment*, **Proc. IEEE**

(2009) Two-Year Killam Research Fellowship

Helioseismology

Several families of “modes”

P– Pressure, P–modes, Acoustic Standing waves
Bounce around surface, $\sim$ 10 *Million* measured

G– Gravity or G–Modes, Buoyancy
Density inversion near top of solar core
No **accepted** G–mode detections in literature

Spherical harmonic description

3 “quantum numbers” for each mode

- n number of radial nodes
- l number of nodes in latitude
- m number of nodes in longitude, $-l \leq m \leq +l$
- Rotational Splitting: each l, n mode has $2l + 1$ “singlets”

Solar Gravity or g–modes

"Holy Grail" of helioseismology — Maximum in the core

Modes ≈ equally spaced in period,

Low frequencies densely packed

Fundamental scaling not quite known

Rotational Splitting by the Solar core: — unknown!

$F_{l,m,n} = f_{nl} - r_{nl} \cdot m$

$+m$ = prograde rotation (so lower observed frequency)

Effective value of r_{nl} depends on rotation profile and mode

(Has to be estimated for each mode.)

May be multiple splittings *inside the Sun*

Additional observed splitting from spacecraft orbit.

G–Modes — Problems

Lots of theory, no repeatable observations

GOLF: low SNR, No spatial resolution:

Get reasonable spatial resolution with Ulysses, (slow)

Limited ($\pm 7^{\circ}.5$) with ACE.

Core rotation: Papers claiming both fast and slow

Theory: G–Mode frequencies depend on:

Irradiance,

Internal solar magnetic fields,

Chemical composition of the Sun,

Stuff we don't know about: WIMPS, monopoles, ???

"Quality factor" $Q \sim 10^{11}$ Interpretated $\Rightarrow$ stable frequencies.

"The hallmark of good science is that it uses models and 'theory' but never believes them." — Martin Wilk

More G–Mode Problems

1) "Quest" for Solar G–Modes started in the 1970's

2) Most data sampled at $\Delta t = 50$ seconds or 1 minute

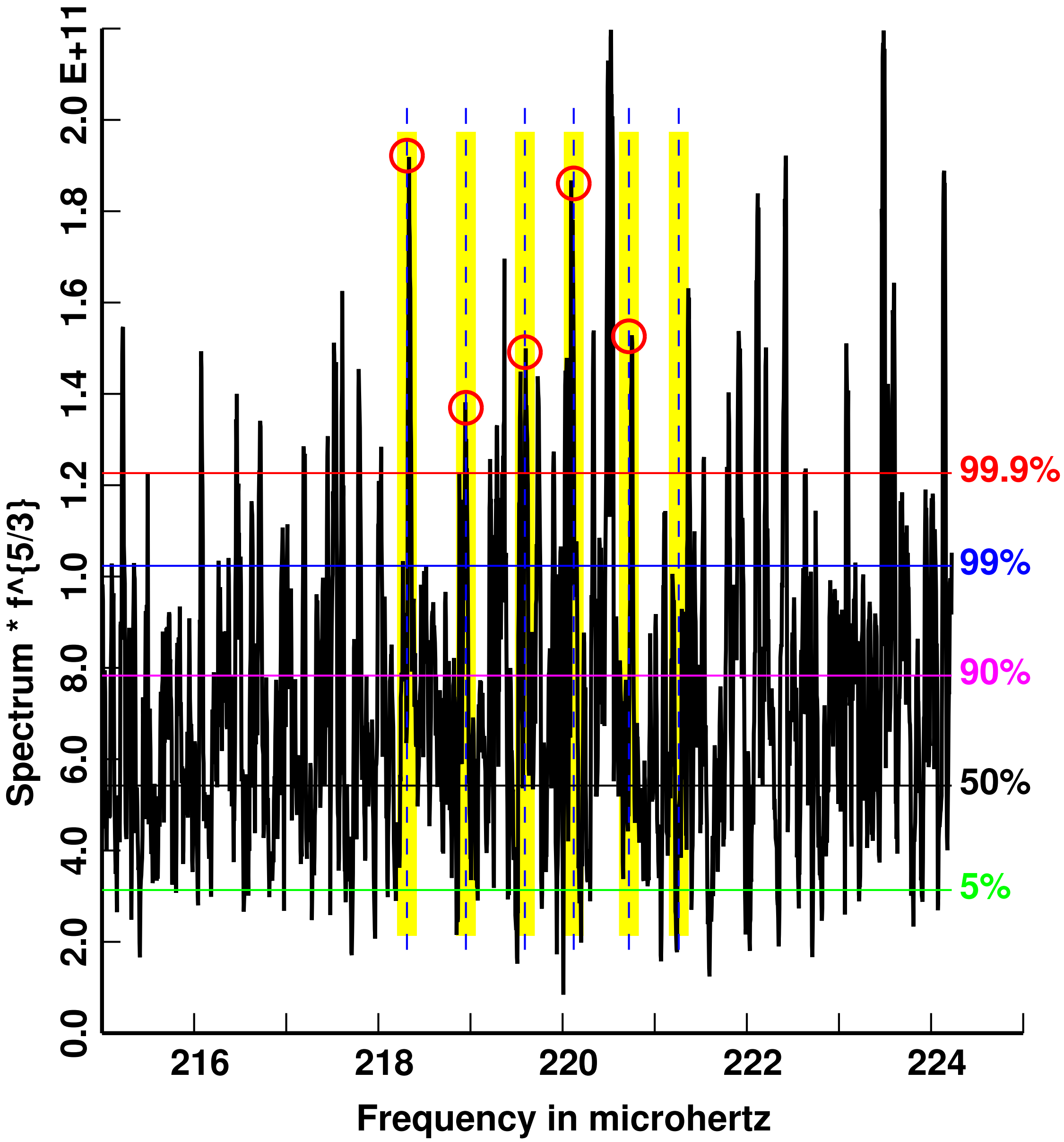
3) Missing data, outliers, spacecraft problems, etc.
   (Must be identified and interpolated)

4) Low pass filter, decimate to 16 minutes for g–modes

6) Power spectra computed from 5 to 10 years of data

- Thousands of G–modes
- *Basic spectra are complicated*
- Splittings give a spectrum that looks like "grass"

Comparison of GOLF and ACE G–Mode Frequencies



Reproducible Detections

Six "candidates"

from SOHO GOLF,

(Not a single mode)

Mathur, **ApJ 668** (2007)

ACE SWEPAM

Proton Density

1998–2005

Frequencies ± 25 nHz

Detections above 99.9%

Five of Six

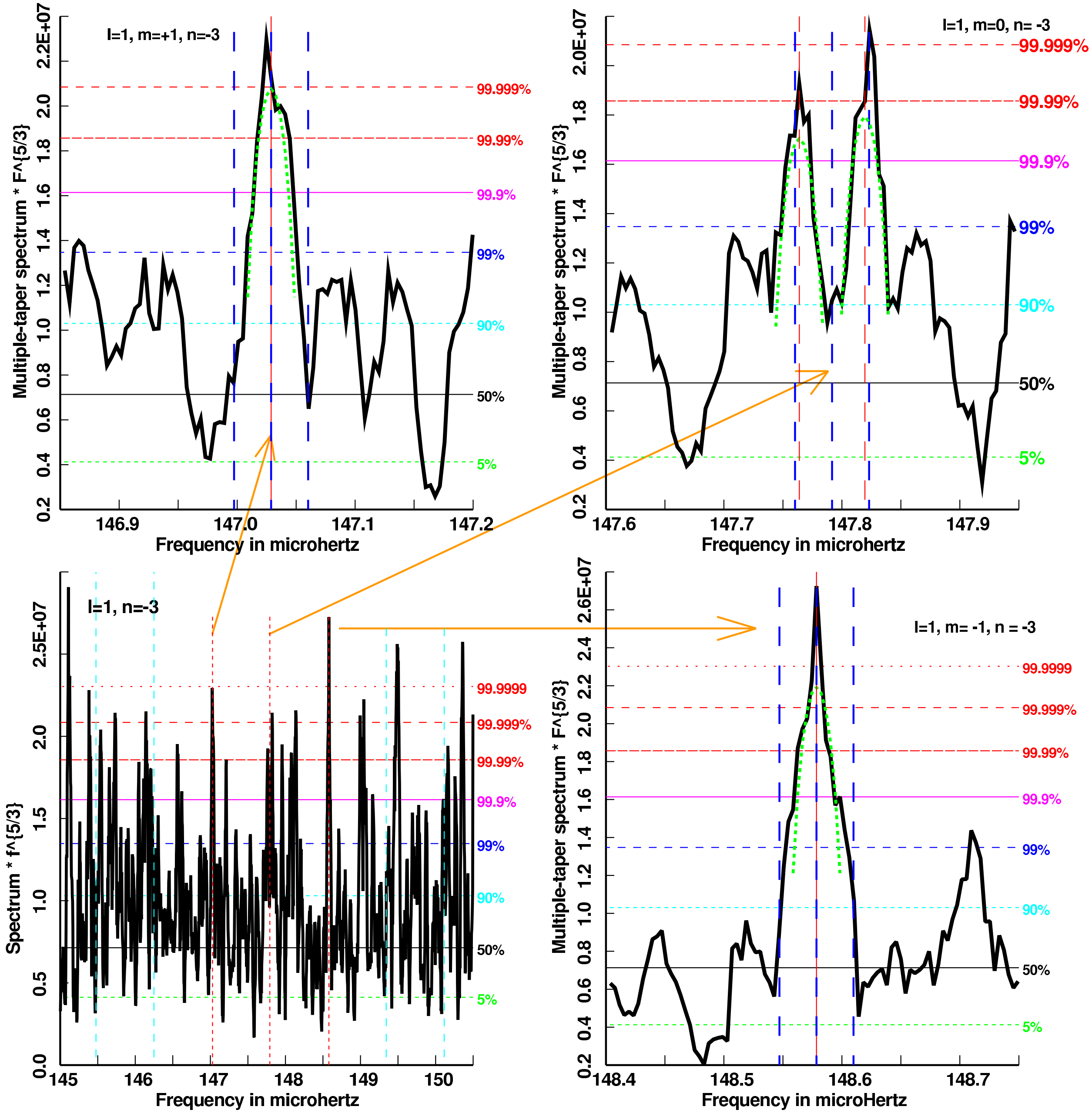
$P = \binom{6}{5}(0.001)^5(0.999)^1$

$\approx 6 \times 10^{-15}$

g–modes are detected

But not identified

Identification of the $l = 1$, $n = -3$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005



Singlet Centers

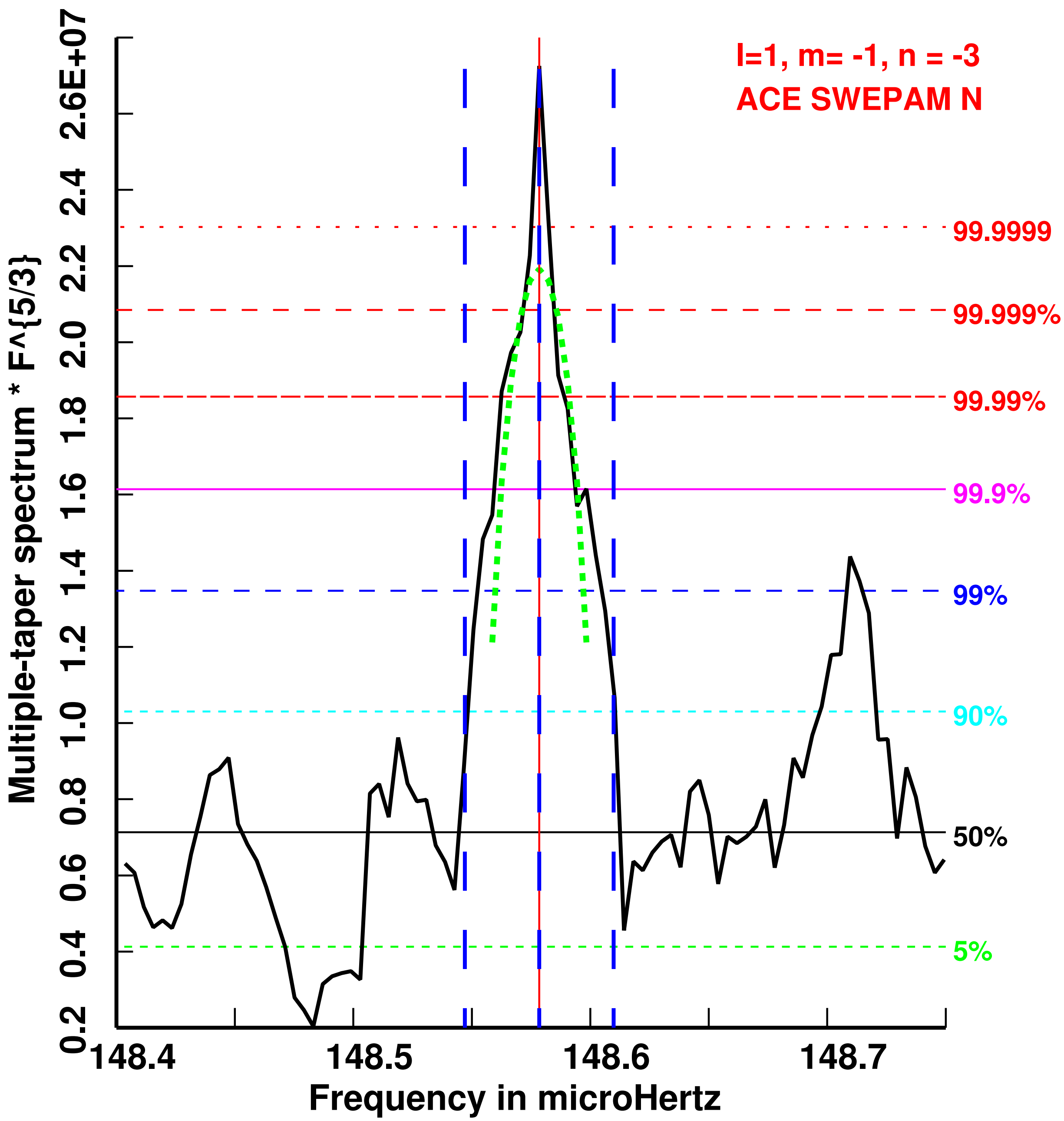
148.578 μHz

147.792

147.029

Identification of the $l = 1$, $n = -3$ G-Mode

ACE Solar wind density, Jan 1998 – Nov 2005



Individual Singlet

$m = -1$

148.578 μHz

Frequency Scale

350 nHz

Spectrum Mostly

5% – 90%

Identification of the $l = 1$, $n = -3$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005

Orbital Splitting, $l = 1$, $m = 0$

1) Mode patterns are $\approx$ spherical harmonics

2) Solar wind flows radially outward

3) $Y_{1,0}(\theta) = c \cdot$ cos(Heliographic co–latitude)
(so zero on the solar equator)

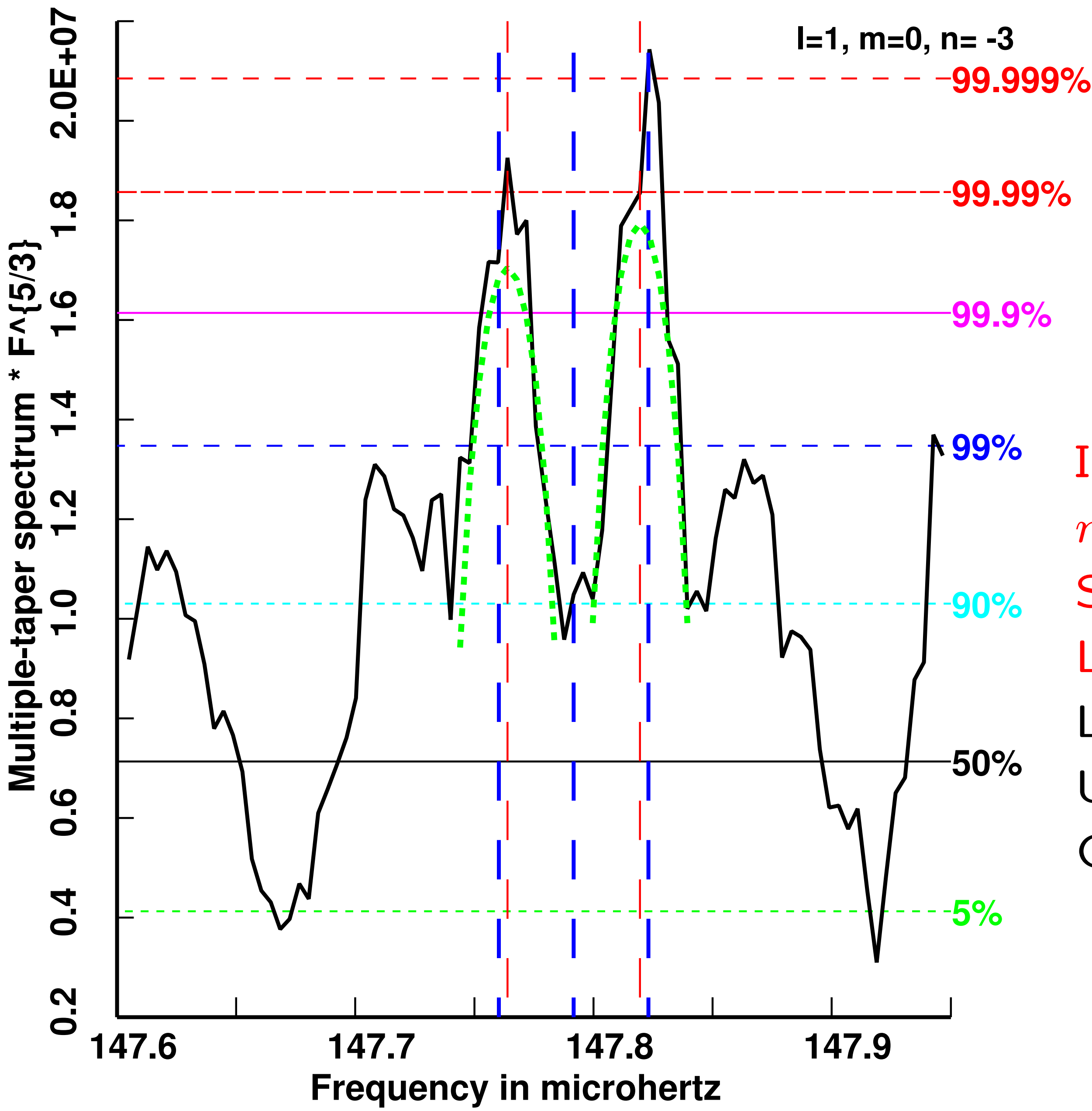
4) Ecliptic tilted $7^\circ.5$ from the solar equator

5) ACE is in the ecliptic at L_1

6) Odd–parity modes split ± 1 cycle/year, low amplitude

7) Phase difference locked to Earth's orbit.
— Don't take too seriously because of the Sun's North–South Asymmetry, ellipticity of orbit.

Identification of the $l = 1$, $n = -3$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005



Individual Singlet

$m = 0$

Split ± 32 nHz

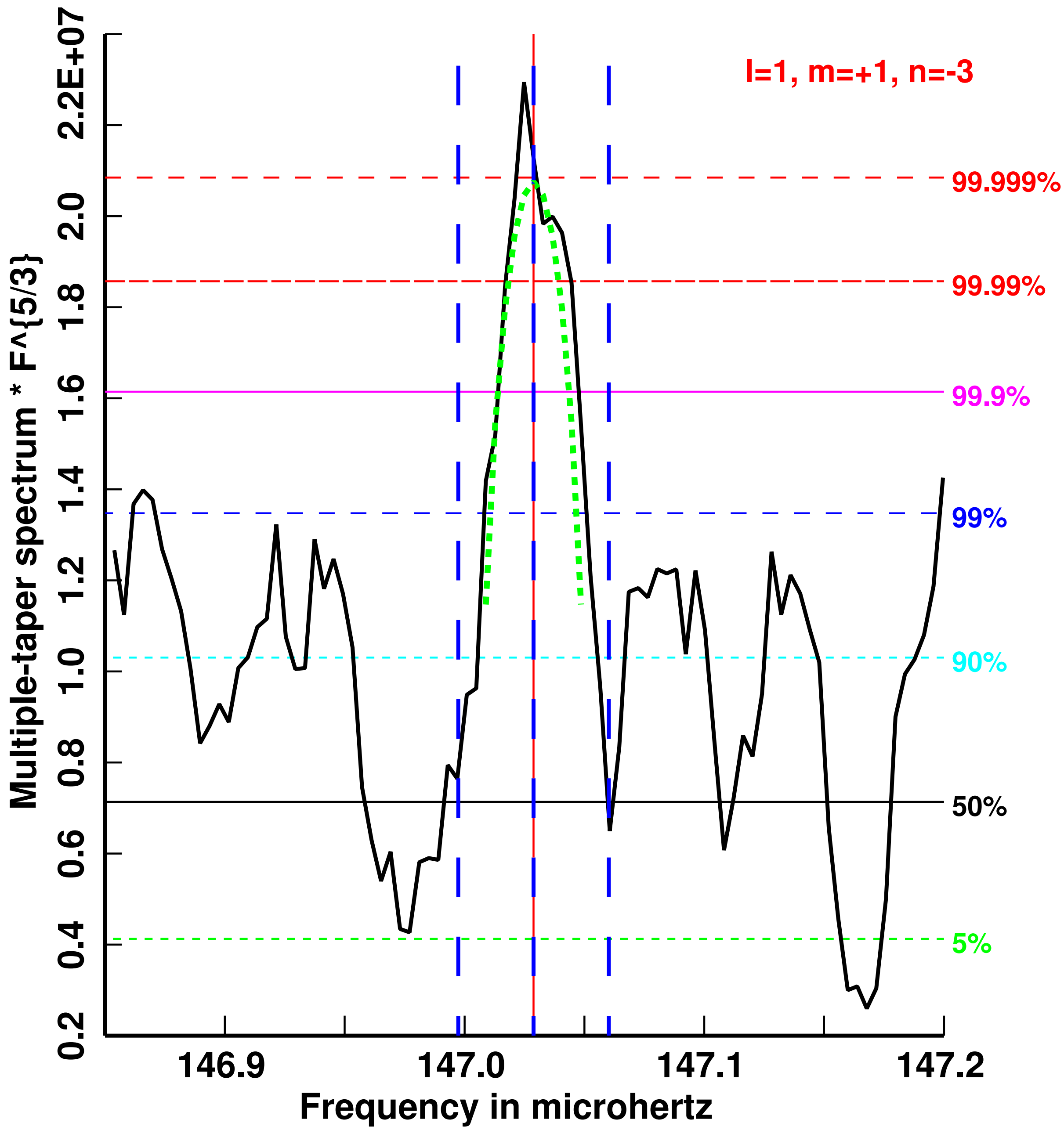
Lower amplitudes

Lower: 147.764 μHz

Upper: 147.820 μHz

Center: 147.792 μHz

Identification of the $l = 1$, $n = -3$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005



Individual Singlet

$m = +1$

147.029 μHz

Identification of the $l = 1$, $n = -3$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005

$l = 1$, $n = -3$ comparisons

147.792 μHz	Estimated Center
150.545 μHz	Gough (1994)
153.300 μHz	Mathur (2007)
153.720 μHz	Provost(2000)

Estimate outside theoretical predictions

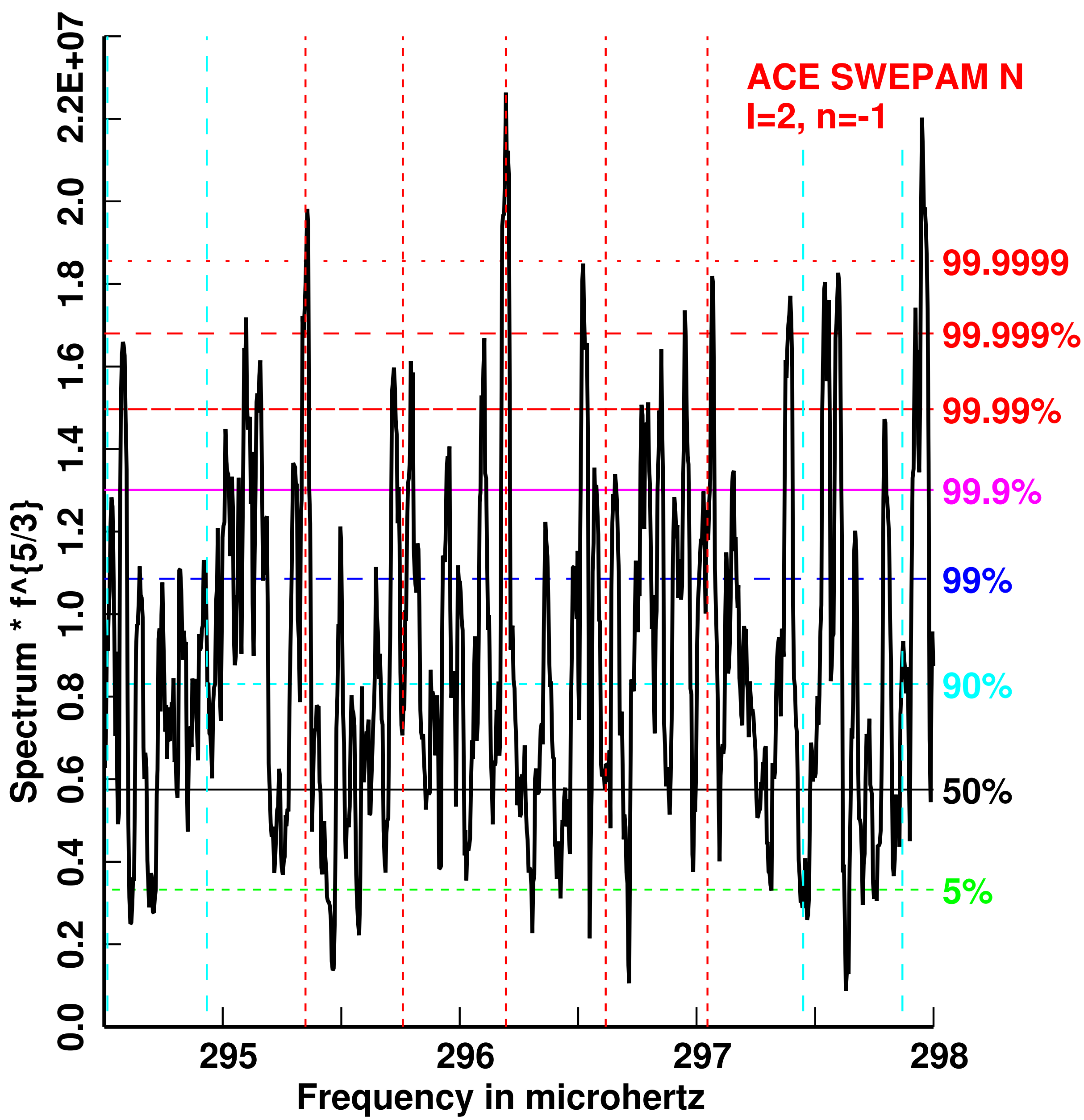
Splitting: (Ω_c = frequency of solar core rotation)

$\Omega_c = 1539$ nHz Estimated

$\Omega_c = 1222$ nHz Gough (1994) — very educated guess!

Effective Core Rotation Period < 7.52 days

Identification of the $l = 2$, $n = -1$ G-Mode
ACE Solar wind density, Jan 1998 – Nov 2005



All 5 (7) singlets
Minimum Prob.
$> 99.9\%$
Center:
296.195μHz
292.53μHz Gough
296.40μHz Mathur
297.01μHz Provost
ALSO:
$l = 3$, $n = 1$
296.883 μHz
296.50 μHz Mathur

Verification from Ulysses

1) Mode identification data from ACE at L_1

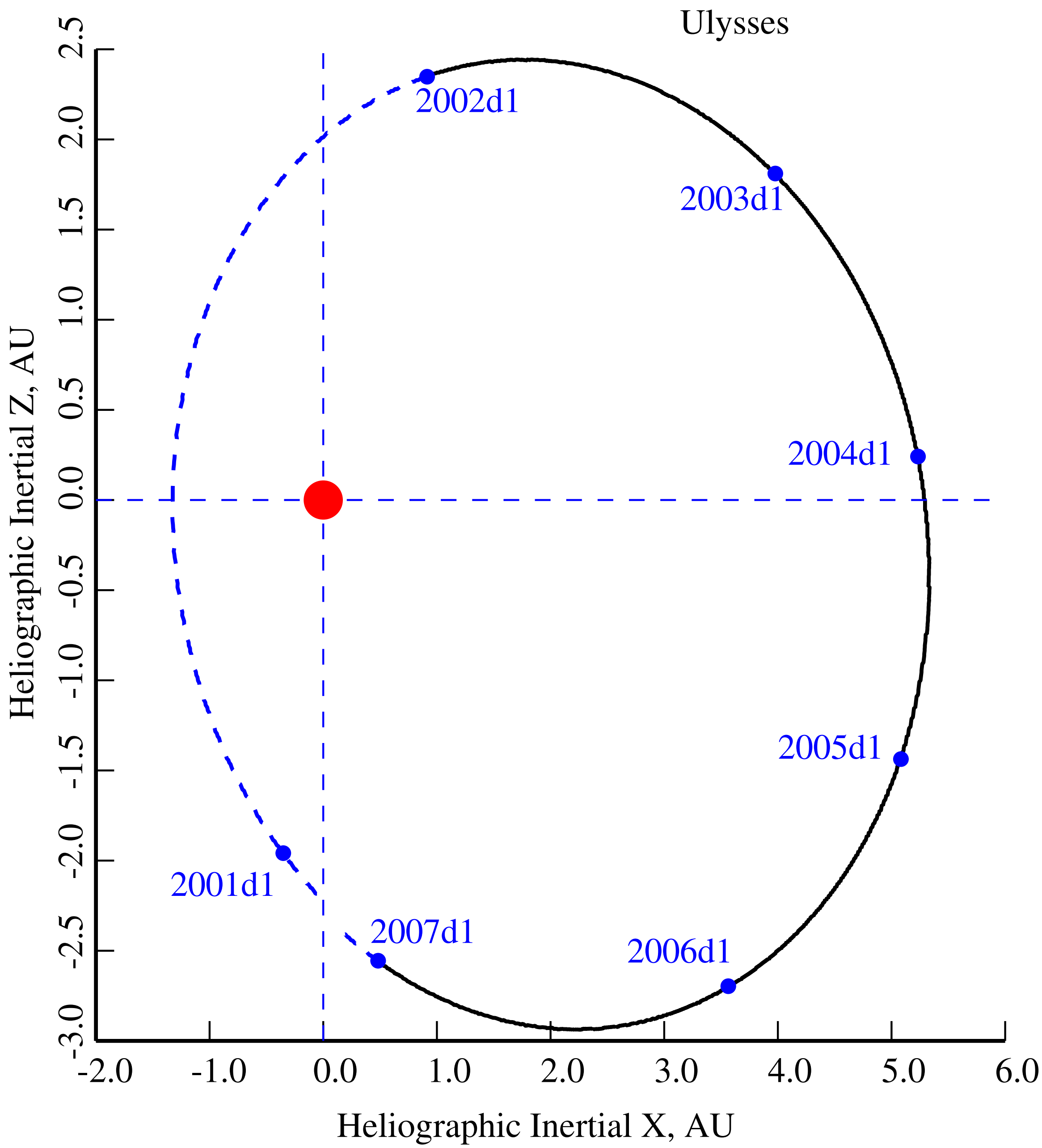
2) Easy to "detect" an $L = 3$ mode in the "wreckage" of an $L = 4$ or higher mode

3) Observed frequencies at ACE are Synodic

4) $f_{ACE}(m) = f_{Sidereal} - m \times 32nHz$
(32nHz = 1 cycle/year, $+m =$ prograde)

5) Ulysses in an approximately sidereal orbit

Ulysses Orbit, Heliographic–Inertial Coordinates



Ulysses Orbit
X—Z plane
(Solar polar)
Variable Radius
≈ quadratic phase
≈ Linear Frequency
2001: Too fast!

Verification from Ulysses

1) Start with synodic frequencies from ACE

2) Offset by $m \times$ 32nHz (needs m correct)

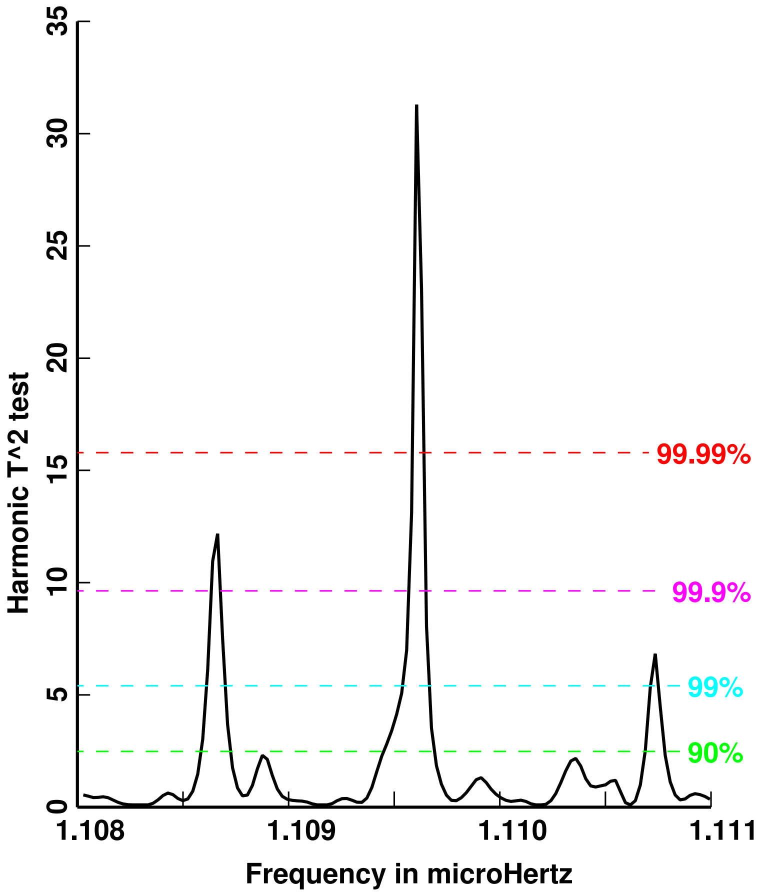
3) Look for a line in spectra of data from Ulysses

4) Some complications
   a) South–to–North pass too fast
   b) Distance to Sun changes from ~ 2 to 5.4 to ~ 2 AU
   approximately quadratic phase (linear frequency) shift
   c) Most modes have 180° phase jumps at nodes

5) Of 12 singlets ($l = 2$, $n = -1$ and $l = 3$,$n = -2$ modes)
   9 matches, 2 misses, 1 iffy (Radial Magnetic Field)

6) Quadratic Phase implies a propagation velocity $\sim c/3$

Ulysses Frequencies in Daily Temperature Spectrum



Multivariate Version

Harmonic–T^2 test

(Common periodicities)

Stockholm, Padua

Daily Data, $>$ 200 years

Frequency *prespecified*

(Ulysses e^-, Nature, 1995)

$\hat{f} \approx 1.10951\,\mu$Hz

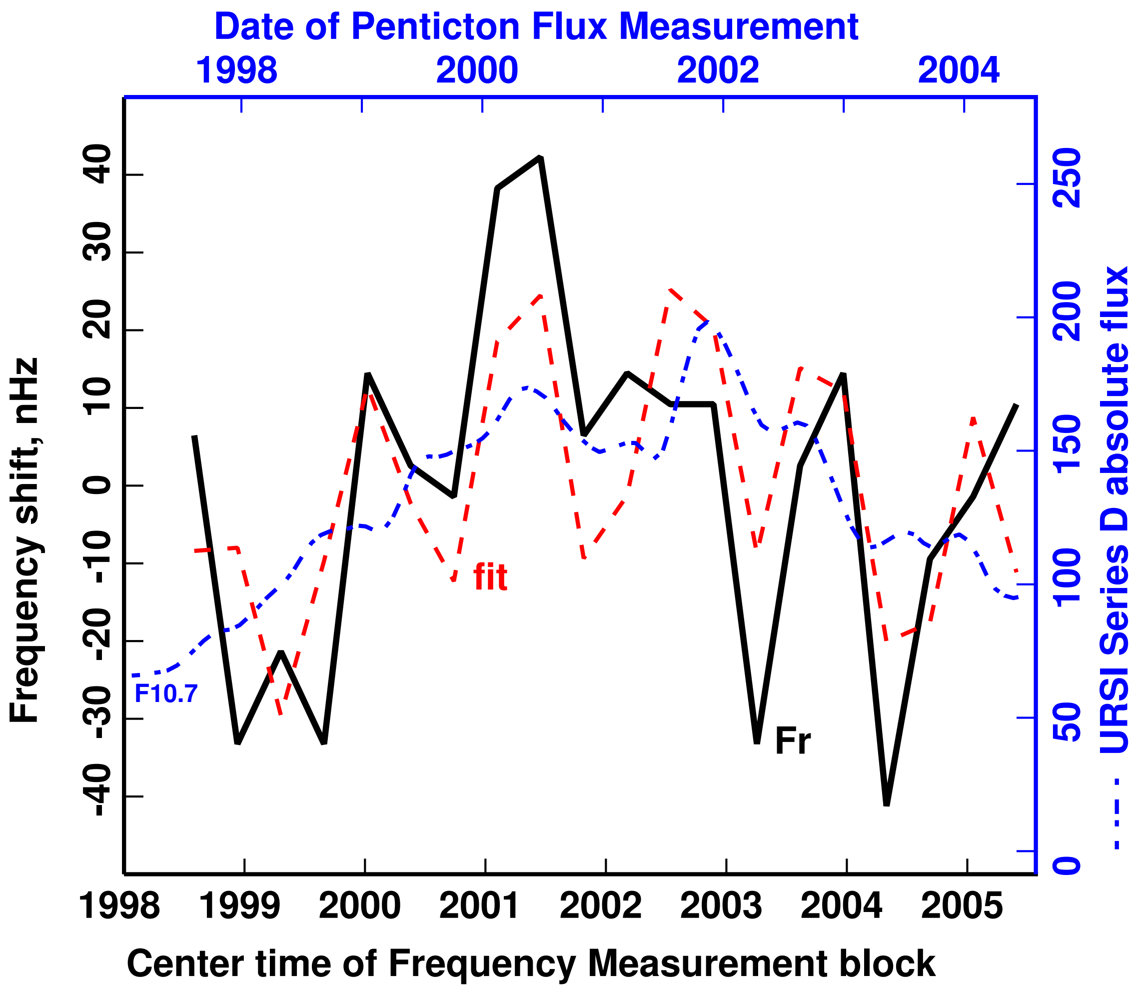
$\doteqdot$ 10.4317 days

Rayleigh Res. 114 pHz

In many sets

Many similar terms

Frequency Shift with Irradiance



ACE Solar Wind
Proton Density
0.97–year blocks
offset 0.359–year
Center: 219.59 μHz
(GOLF "candidate")
Fit:
1.276–year "Tachocline"
$F_{10.7}$ solar flux
Significant Shift
$\sim \pm 20$ nHz
Mode frequency
LEADS
$F_{10.7}$ by ~ 1 year

What went wrong — why weren't g–modes detected?

1) **Wrong** observing frequency (optical)
   Low signal–to–noise ratio.

2) **Poor analysis methods**, poor interpolation through gaps
   Spectra are mostly periodograms — lines too narrow,
   poor sensitivity, high false detection rate.

4) **Taking theory too seriously**.
   Assuming "high Q" implies high frequency stability
   Theory predicts rapid attenuation with l.

5) **Not taking theory seriously enough**
   Theory predicts frequency shift with irradiance,
   magnetic fields (Observed in P–modes!)

6) **"Not invented here"** attitude,
   Frequent "reinvention" of analysis methods.

7) Not enough **"exploratory data analysis"**

Solar Modes in Space: Conclusions

- Same frequencies from optical helioseismology data from SOHO and various instruments on ACE.
- Frequency modulation: Solar (22–year Hale) cycle
  Shift in f leads irradiance by ~ 1 year
  Mode Splitting from Orbit
- Some G–Modes Identified, all $2l+1$ singlets
  Repeatable Detections
- Tentative detection of some modes with $l > 12$
- Agreement with theory:
  - Good at higher ($\sim 250\,\mu$Hz) frequencies
  - Less at lower ($\sim 150\mu$Hz) frequencies
- The Sun has a FAST core, $> 3\times$ surface rate.

Overall Summary

Multitapers: A better way to analyze time series
(Power Spectra, autocorrelations, lots of additional tools)
F–test for periodic components

Climate:
CO_2 effects too strong to miss
Complicated stochastic structure

Solar Gravity Modes:
Detected and Identified
Repeatable Detections
The Sun has a fast core!

Thank you!