Lecture 1.

classical: $\underline{a}$ simple Lie alg, rep theory Verma modules f.d. reps BGG categ $\mathcal{O}$ - soc the central characters. etc - next lectures we'll see how much of this goes over, what mod's are necessary etc in affine, more gen. cases
→ fix some overall notations

$\underline{a}$ Lie-alg, $\mathcal{U}(a)$ univ. env.-alg

$\mathcal{U}(a) = T(a)/ a\otimes b - b\otimes a - [a,b]$

$T(a) = \bigoplus_{n\geq 0} T^n(\underline{a})$ $\mathbb{Z}_+$-graded, relations defining $\mathcal{U}(a)$ are not graded but filtered by $n\in\mathbb{Z}_+$
$\mathcal{U}^n(a) = \text{im of } \bigoplus_{m\leq n} T^m(a)$. fil

PBW: associated graded alg of $\mathcal{U}(\underline{a})$ is $S(\underline{a})$

PBW basis: pick an ordered basis of $\underline{a}$ then $\mathcal{U}(\underline{a})$ has a basis of ordered monomials

$\underline{a}$ f.d. $\Rightarrow$ $\mathcal{U}(\underline{a})$ noetherian

M - module $\mathcal{U}(a)$ $\Rightarrow$ M has an max. submodule

$\underline{a}$ simple - no nontrivial ideals killing form of $\underline{a}$ $K(x,y) = \mathrm{tr}\, \mathrm{ad}\,x\, \mathrm{ad}\,y$

– ~~$\underline{a}$ of~~ a ~~dim of~~ f.d. rep of $\underline{a}$ on V, rep of $U(\underline{a})$ on V

$f : \underline{a} \to \mathrm{end}(V)$

basic example: sl_2 x, y, h $[h,x] = 2x$

$[h,y] = -2y$

$[x,y] = h$

$K(x,y) = 4, \quad K(h,h) = 8$ non-deg. form

Casimir elt $\Omega = \frac{h^2}{8} + \frac{xy}{4} + \frac{yx}{4}$

$\Omega \in U(sl_2)$

$= \frac{h(h+2)}{8} + \frac{yx}{2}$.

$Z(U(\underline{a})) = Z(\underline{a})$

$\Omega \in Z(\underline{a})$ $\quad Z(\underline{a}) \simeq \mathbb{C}[\Omega]$

understand f.d. reps of $sl_2/\mathbb{C}$.

V $\dim V = n+1$ $\quad h : V \to V$ $\quad hv_0 = \mu v_0$

$\mu \in \mathbb{C}$

~~xv_0 eigenval~~ $\mu+2$ $\quad x$

$hxv_0 = xhv_0 + 2xv_0 = (\mu+2)xv_0$

$\Rightarrow \exists\ r$ s.t. $x^{r-1}v_0 \neq 0$ $\quad x^r v_0 = 0$ $\quad v_0 = x^{r-1}v$.

$xv_0 = 0$, $\mu + 2r = \lambda$ PBW: $\quad \not\subset$

check: $\{v_0, yv_0, \ldots, y^s v_0\}$ eigenvalues

are $\lambda, \lambda-2, \ldots \lambda - 2s$. so $\exists n$ $y^{n+1}v_0 = 0$

and this is a submod $\Rightarrow \lambda = n$, $xy^{n+1}v_0 = 0$

$= y^n(h-n)v_0 = 0 \Rightarrow h = n$

i.e. V has a basis ~~v_0~~ $\{v_0, \dots v_n\}$

~~x~~ y

$$x v_i = (n-i+1) v_{i-1} \qquad h v_i = n v_i$$

$$y v_i = (i+1) v_{i+1} \qquad v_{-1} = 0, \quad v_{n+1} = 0$$

easily checked this is an action.

more is true. $x v_i = (\lambda - i + 1) v_{i-1} \qquad h v_i = \lambda v_i$

$$y v_i = (i+1) v_{i+1} \quad , \quad v_{-1} = 0$$

this also defines an action of $\mathfrak{sl}_2$ on an inf. dim space $M(\lambda)$ - Verma module.

analyze $M(\lambda)$: ~~$xy v_i = (i+1)(\lambda - i) v_i$~~

~~$x v_{i}$~~ $\lambda \notin \mathbb{Z}_+ \Rightarrow M(\lambda)$ irr.

submod. N $h: N \to N$ N contains an eigenvector of h $\Rightarrow$ $v_i \in N$ for some i

$\Rightarrow v_k \in N \quad \forall\, k \geq i \qquad x v_i = (\lambda - i + 1) v_{i-1} \in N$

and $\lambda - i + 1 \neq 0 \Rightarrow v_{i-1} \in N$ —

$M(\mu)$ has a JH series

$\lambda \in \mathbb{Z}_+$ $0 \to M(-\lambda-2) \to M(\mu) \to V(\mu) \to 0$

$\lambda = \mu$

$$\Omega v_0 = \left(\frac{h(h+2)}{8} + \frac{yx}{2} \right) v_0 = \frac{\lambda(\lambda+2)}{8}$$

~~Ω . M' $\lambda \notin \mathbb{Z}_+$~~

Ω acts on $M(\lambda)$ and $M(-\lambda-2)$ by the same value. and ~~$\mu \neq \lambda$~~ $\mu \neq -\lambda-2$

~~then~~ $\to$ Ω acts on $M(\lambda)$ by $M(\mu)$ by diff scalars.

M for sl_2. $M = \bigoplus_{\mu \in \mathbb{C}} M_\mu$ eigensp

$M_\mu = \{ m : h\, m = \mu m \}$ dim $M_\mu < \infty$

$\Omega : M \to M$. $\Omega : M_\mu \to M_\mu$

$$M_\mu = \bigoplus (M_\mu)_\chi = \{ m \in M_\mu \; (\Omega - \chi)^r m = 0, \; r \geq 0 \}$$

$$M_\chi = \bigoplus_{\mu} M_{\mu, \chi}$$ $\mathfrak{g}$-submodule.

$$M = \bigoplus_{\chi} M_\chi$$

$M = M_\chi$ irreducible f.d. modules with this property? f.d. ~~ok~~

$\chi = \frac{\lambda(\lambda+2)}{8}$ $\quad M(\lambda), M(-\lambda-2)$

$\lambda \notin \mathbb{Z}_+$

$\lambda \in \mathbb{Z}_+$ $\quad V(n), M(-n-2)$ $\quad$ examples

and ~~given~~ any

$M = M_\chi$, eigenvalues $\subseteq \mu - 2\mathbb{Z}_+$

~~then~~ M irr $\Rightarrow M$ one of the above.

Weyl's th

$\underline{g}$ simple Lie alg. $\quad \mathfrak{h} \hookrightarrow \underline{g}$ $\quad$ CSA

elements of $\mathfrak{h}$ act semisimply on $\underline{g}$ thro the adj

and $\mathfrak{h}$ max with this prop. $\mathfrak{h}$ abelian

$$\underline{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} g_\alpha.$$

$R \subseteq \mathfrak{h}^*$ finite, $0 \notin R$.

$$\underline{g}_\alpha = \{x \in \underline{g} : [h,x] = \alpha(h)\, x\}.$$

$K|_{\mathfrak{h}\times\mathfrak{h}}$ non-deg. R root system.

and the def. on a real subsp. of $\mathfrak{h}^*$.

$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \subseteq \mathrm{Aut}\, \mathfrak{h}^*_{\mathbb{R}}$ weyl gp.

simple system Π $\{\alpha_1, \dots \alpha_n\}$

$\alpha \in R \Rightarrow \alpha = \sum n_i \alpha_i$ all $n_i \in \mathbb{Z}_+$
$n_i \in \mathbb{Z}_-$

$R = R^+ \cup R^-$ $\alpha \in R^+$ $x_\alpha^+, x_\alpha^-, h_\alpha$ chevalley basis
$[x_\alpha^+, x_\alpha^-] = h_\alpha$

$\mathfrak{sl}_{n+1}(\mathbb{C})$ $\varepsilon_i : \mathbb{C}^{n+1} \to \mathbb{C}$ i^{th} coord.

$\mathfrak{h}$ = diag. matrices $\subseteq \mathbb{C}^{n+1}$ $\{\varepsilon_i - \varepsilon_j, \ 1 \le i \ne j \le n+1\} = R$

$\{\varepsilon_i - \varepsilon_{i+1}\}_{1 \le i \le n} = \Pi$.

$\Omega \in \mathfrak{z}(\mathfrak{g})$.
but this is now only a small part of the center.

V f.d. rep of $\mathfrak{g}$ $h_1, \dots, h_n$ commuting operators on V common eigenvector. $\lambda_1 \dots \lambda_n$ are eigenvalue $\mu \in \mathfrak{h}^*$ $\mu(h_i) = \lambda_i$

$h_i v = \mu(h_i) v$.

$\alpha \in R^+$ $\quad h_i x_\alpha v = (\mu+\alpha)(h_i) x_\alpha v$

$x_{g_1}^{r_1} x_{g_2}^{r_2} \cdots x_{g_k}^{r_k} v \ldots \quad (\mu + \sum r_i \alpha_i) \quad g^{\alpha}$

$\alpha_1 \ldots \alpha_n$ are lin. ind so eigenvalues ...

$\Rightarrow \exists \ v_0 \in V$ s.t. $h v_0 = \lambda(h) v_0$

$x_\alpha v_0 = 0$

$n^- = \bigoplus_{\alpha \in R^-} g_\alpha$.

~~PBW = $U(g) v_0$ =~~ $U(n^-) U(h) U(n^+) v_0$.

$= \underline{U(n^-)} v_0$.

$\{v_0, y_i v_0, \ldots y_i^k v_0\}$ sl_2 submod

$y_i^{k+1} v_0 = 0$

$\Rightarrow k = \lambda(h_i)$

$\Rightarrow$ ~~V~~ $\longrightarrow$ λ $\quad \lambda(h_i) \in \mathbb{Z}_+$, $V_\lambda \neq 0$

$x_\alpha V_\lambda = 0 \ \forall \alpha \in R^+$

... is unique

Converse is also true $\lambda(h_i) \in \mathbb{Z}_+$ $\exists!$ (up to iso)

irr. mod. V s.t. $V_\lambda \neq 0$ $\quad x_\alpha V_\lambda = 0$.

$P^+ = \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}_+\}$

~~we~~

Sorry: V is generated by an element v_λ with rel'ns: $h v_\lambda = \lambda(h) v_\lambda$

$$x_\alpha^+ v_\lambda = 0$$

$$(x_\alpha^-)^{\lambda(h_\alpha)+1} v_\lambda = 0$$

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natural basis for $V(\lambda)$, explicit action of $\underline{g}$ on $V(\lambda)$

canonical bases / global crystal basis / crystal bases are about

simpler questions we can ask. - what is the character for instance?

wt module: $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ modules on which $\mathfrak{h}$ acts semisimply

$M_\mu = \{m \in M : hm = \mu(h) m \ \forall h \in \mathfrak{h}\}$

all modules we have seen so far are wt modules

$ch_M : \mathfrak{h}^* \to \mathbb{Z}_+ \qquad \mu \to \dim M_\mu.$

ch

ch_M tells you what M looks like as an $\mathfrak{h}$-mod.

$= \cancel{\sum \operatorname{tr} \rho(h) V_\lambda}$

weyl character formula $\quad ch\, V(\lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim V(\lambda)_\mu e(\mu)$

<u>Prop.</u> $\nu \in \mathfrak{h}^*$, $w \in W$ $\quad \dim V(\lambda)_\nu = \dim V(\lambda)_{w\nu}$.

W invariant

$\operatorname{tr} h = \sum \mu(h) \dim V(\lambda)_\mu$.

$(\operatorname{tr} h)_\mu =$

naturally to study W invariant poly. fns.

on $\mathfrak{h}^*$ a.

$U(\mathfrak{h}) \simeq S(\mathfrak{h}) \simeq$ poly fns on $\mathfrak{h}^*$

$\mathfrak{h}$ abelian

$$ch\, V(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e(w(\lambda+\rho))}{\sum \epsilon(w) e(w\rho)} \in \mathbb{Z}[P]$$

$\mathbb{Z}[P] =$ $\mathbb{Z}$-valued fns on P, zero outside a finite set . algebra. via convolution.

ingredients that go into proving the weyl character formula

- verma modules.

$\lambda \in \mathfrak{h}^*$ $\quad M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$

$\mathbb{C}_\lambda$ 1-dim $\mathfrak{b}$-mod $\quad \mathfrak{n}^+ \mathbb{C}_\lambda = 0$

$h v_\lambda = \lambda(h) v_\lambda$

$M(\lambda)$ free $U(\mathfrak{n}^-)$-mod, rank 1.

wt $M(\lambda) \subseteq \lambda - Q^+$ $\quad Q^+ = \cancel{\text{span of } R^+} = \mathbb{Z}_+ \text{ span of } \Pi$

$\dim M(\lambda)_\lambda = 1$

$M(\lambda)$ has a ! irr. quot. $V(\lambda)$

$M(\lambda)$ universal prop with resp to being gen. by v_λ with rel^{n} above

<u>Compare with $\mathfrak{sl}_2$:</u> when is $M(\lambda)$ irr. ?

<u>known.</u> $\lambda(h_i) \notin \{0, 1, 2, \dots\}$

Jordan Hölder series of $M(\lambda)$?

$\mathcal{U}(a)$ noetherian

$M(\lambda)$ irr ok $M_0 = M(\lambda) \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_r \supseteq \dots$

M_0/M_1 irr $\cong V(\lambda)$ $\qquad M_1/M_2$ irr

why does this stop?

$0 \leftarrow V(n) \leftarrow M(n) \leftarrow M(-n-2) \leftarrow 0$

to answer this one needs to

known something about these irr. quotients

that occur.

ex. ~~wt M_i~~

- Each M_i is an $\mathfrak{h}$-submodule and $\mathfrak{h}$ acts $(M_i)_\mu = M_i \cap M_\mu$.

$$M_i = \bigoplus_{\mu \in \mathfrak{h}^*} (M_i)_\mu. \quad M_i \text{ wt module.}$$

$$\text{wt } M_i \subseteq \text{wt } M \subseteq \lambda - Q^+$$

- M_i/M_{i+1} irr and wts $\subseteq \lambda - Q^+$

$\Rightarrow M_i/M_{i+1}$ gen. by an elem v_{μ} w/ $h(v_\mu)$, $n^+ v_\mu = 0$

$\Rightarrow$ M_i/M_{i+1} is iso to the irr. quot $V(\mu_i)$ of $M(\mu_i)$

So now at least we know that sub irr. quot are $V(\mu_i)$ — but this could still be too many.

- $\mathfrak{z}(\mathfrak{g})$ $u \in \mathfrak{z}(\mathfrak{g})$ ~~u act~~

$$M(\lambda) = U(\mathfrak{g}) m_\lambda$$

$$z.m = z(g m_\lambda) = g(z m_\lambda)$$

$$z m_\lambda \in M(\lambda)_\lambda \qquad \dim M(\lambda)_\lambda = 1$$

$$z m_\lambda = \chi_\lambda(z) m_\lambda \qquad \chi_\lambda(z) \in \mathbb{C}$$

ex: $\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is a hom. of algebras.

$M_i \subseteq M$ $\mathfrak{z}(\mathfrak{g})$ acts on M_i via χ_λ

and hence also on M_i/M_{i+1} as χ_μ.

on the other hand

$$\frac{M_i}{M_{i+1}} \simeq V(\mu_i) \leftarrow M(\mu_i)$$

$$\mathfrak{z}(\mathfrak{g}) \circlearrowright \quad \chi_{\mu_i}$$

ques. becomes: when is $\chi_\lambda = \chi_\mu$?

famous thm of HC: $\chi_\lambda = \chi_\mu \Leftrightarrow \lambda = \omega(\mu+\rho)-\rho$ for some $w \in W$

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ $\quad Z(g) \simeq U(\mathfrak{h})^W$

Cor: Verma modules have finite comp. series

$$ch\, M(\lambda) = \sum_{\lambda - w\lambda \in Q^+} c_{w\lambda}\, ch\, V(w(\lambda+\rho)-\rho)$$

$c_{w_\lambda} = \# [M(\lambda) : V(w(\lambda+\rho)-\rho)]$

$$ch\, V(\lambda) = \sum b_{w_\lambda}\, ch\, M(w(\lambda+\rho)-\rho)$$

ch $M(\mu)$ are easily written down

w freeness of HC

$\lambda \in P^+$ W-invariance to deduce weyl char form!

$y, w \in W$ $$ch(M(y \circ \lambda)) = \sum c^\lambda_{y,w}\, V(w \cdot \lambda)$$

what are these nos ?

Kazhdan-Lusztig theorem $\exists$ poly. $P^\lambda_{y,w} \in \mathbb{Z}_+[q]$

$P_{y,w}(1) = c^\lambda_{y,w}$

$\mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{h})$ alg. hom HC hom.

$\mathcal{O}$ BGG. $M = \bigoplus_{\chi: \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}} M_\chi$

$\chi: \mathfrak{z}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W \to U(\mathfrak{h})$

~~M~~ ~~M~~.

any irr. mod in M ~~has~~ $V(\lambda)$

$M = \bigoplus_{\lambda \in \mathfrak{h}^*/W} M_{\chi_{\cancel{\lambda}}}$

enough to underst. $M_{\chi \cdot \lambda \cdot}$

$\to$ Soergel,