Lecture 2

$\mathfrak{g}$ simple Lie alg, $\mathfrak{h} \hookrightarrow \mathfrak{g}$ CSA $R, \Pi, R^+ \subseteq \mathfrak{h}^*$

$Q =$ $\mathbb{Z}$-span of R, $P = \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}\}$

here we fix a Chevalley basis

$\{x_\alpha : \alpha \in R\} \cup \{h_i : 1 \le i \le n\}$ basis of $\mathfrak{g}$

W - weyl gp $\theta \in R^+$ highest root

i.e. unique element in R^+ s.t. $\theta - \alpha \in Q^+$

for all $\alpha \in R^+$

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$$

$$\lambda \in \mathfrak{h}^* \qquad M(\lambda) = \mathcal{U}(\mathfrak{g}) \Big/ \begin{matrix}\text{left ideal} \\ \mathfrak{n}^+ \\ h - \lambda(h)\end{matrix}$$

$V(\lambda)$ unique irr. quotient

defn: M be a $\mathfrak{g}$-module, M is a wt module if $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, M_μ called weight space.

$\mathrm{wt}(M) = \{\mu \in \mathfrak{h}^* : M_\mu \neq 0\}$

$M(\lambda)$, $V(\lambda)$ are wt. modules

wt $V(\lambda) \subseteq$ wt $M(\lambda) \subseteq \lambda - Q^+$.

V any wt-module for $\underline{g}$

$ch_V : \mathfrak{h}^* \to \mathbb{Z}$, $\mu \to \dim V_\mu$.

V is f.d. then ch_V has finite support.

$\mathcal{F}$ = functions for $\mathfrak{h}^* \to \mathbb{Z}$ whose support lies in a finite union of cones

$\lambda_s - Q^+ \; : 1 \le s \le N$.

$\mathcal{F}$ is a ring $(f * g)(\mu) = \sum f(\lambda) g(\nu)$

$e(\lambda) \in \mathcal{F}$ char. fn $e(\lambda)(\lambda) = 1$ $\quad \mu = \lambda + \nu$

$(\mu) = 0 \; \mu \neq \lambda$

$$M(\lambda) = \bigoplus_{\eta \in Q^+} M(\lambda)_{\lambda - \eta} \qquad \text{wt } M(\lambda) = \lambda - Q^+$$

$$\dim M(\lambda)_{\lambda-\eta} = \dim U(n^-)_{\lambda - \eta} < \infty \text{ by PBW}$$

$$ch\, M(\lambda) = \frac{e(\lambda)}{\prod_{\alpha \in \Phi^+} (1 - e(-\alpha))} \qquad \rho = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha$$

weyl character formula.

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w\in W} \epsilon(w) e(w(\lambda+\rho))}{\sum_{w\in W} \epsilon(w) e(w\rho)}$$

to get weyl character formula.

① $M(\lambda)$ has JH series and JH components $V(w(\lambda+\rho)-\rho)$.
V wt module and.

② V f.d. $\Rightarrow$ $\dim V_\mu = \dim V_{w\mu}$ $\forall \mu \in \mathfrak{h}^*$

$y \in W$ $\quad M(y\circ\lambda) \supseteq M_1 \supseteq \ldots$

$$\operatorname{ch} M(y\circ\lambda) = \sum_{w\circ\lambda \leq y\circ\lambda} c^\lambda_{y,w} \operatorname{ch} V(w\circ\lambda)$$

c^λ_{yw} $\quad |W|\times|W|$ matrix upper triang

1 on diag

$$c^\lambda_{y,w} = [M(y\circ\lambda), V(y\circ\lambda)] = 1.$$

invert matrix $\operatorname{ch} V(w\circ\lambda) = \sum b^\lambda_{wy} \operatorname{ch} M(y\circ\lambda)$

use W-inv. to get ch. for.

Conclude our discussion of s.s. case with 2 imp. results

① Weyl's thm: finite dimensional reps of s.s. Lie alg are comp reducible.

② $\mathcal{O}$ BGG f.g. $\underline{g}$-mod, $\underset{K}{\mathcal{M}}$

M wt modules $wt(M) \subseteq \bigcup_{s=1}^{k} \lambda_s - Q^+$

$\lambda_1, .. \lambda_s \in \mathfrak{h}^*$ depending on M.

$M(\lambda) \in \mathcal{O}$

irr. objects in $\mathcal{O}$ are $V(\lambda)$ $\lambda \in \mathfrak{h}^*$

$\mathcal{O}$ is not a semisimple categ i.e. $\exists$ indecomposable reducible objects in $\mathcal{O}$.

$M \in \mathcal{O}$ $\quad$ ~~$M = \bigoplus_{\chi \in Hom(Z(g))} M^\chi$~~

$\chi : Z(g) \to \mathbb{C}$

$M^\chi = \{ m \in M : (z - \chi(z))^N \cdot m) = 0 \}$

$M = \bigoplus_{\chi \in \mathfrak{h}^*} M_\chi$ $\quad$ $\mathcal{O}_\chi$ full subcateg. of mod M s.t. $M = M_\chi$

$\cancel{\ast}$ $M(\lambda) \in \mathcal{O}_\chi \Leftrightarrow \chi = \chi_\lambda$.

$\mathcal{O} = \bigoplus \mathcal{O}_{\chi_\lambda}$ $\quad \lambda \sim \cdot \mu \Leftrightarrow \chi_\lambda = \chi_\mu$.

$\lambda \in \mathfrak{h}^*/W$ $\quad \mathcal{O}_\chi$ has only finitely many simple objects

Study of $\mathcal{O}_\chi$ - methods ~~of~~ of Soergel etc.

~~ref~~ [CPS]

$\boxed{\text{IMP}}$ center is crucial.

$U(\mathfrak{g})$ noetherian used crucially.

<u>affine case:</u> $\mathfrak{g}$ $\quad L(\mathfrak{g})$ = loop alg

$\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$. $= \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n,-m} K(x,y)c$.

c central, $[d, x \otimes t^n] = n x \otimes t^n$.

$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$

KM alg so it comes equipped with $\hat{R}, \hat{R}^+, \hat{\Pi}$

$\hat{W}$ = affine weyl grp

describe $\hat{R}$ explicitly

$\delta \in \hat{\mathfrak{h}}^*$ $\quad \delta(d) = 1$ $\quad \delta|_{\mathfrak{h}\oplus\mathbb{C}c} = 0$

roots are eigenvalues for action of $\hat{\mathfrak{h}}$.

$$\mathfrak{g}_\alpha \otimes t^n \qquad \alpha + n\delta \qquad \begin{matrix}\alpha \in R \\ n \in \mathbb{Z}\end{matrix}$$

$$h \otimes t^n \qquad n\delta$$

$\hat{R} = \{\alpha + n\delta : \alpha \in R,\ n \in \mathbb{Z}\} \cup \{n\delta : n \neq 0\}$

$\hat{\Pi} = \{\alpha_1, \ldots, \alpha_n, \underbrace{-\theta + \delta}_{\alpha_0}\}$

root ~~vector~~ $\mathfrak{g}$ $\quad \mathfrak{n}$

$\hat{R}^+ = \{\alpha + n\delta : \alpha \in R^+, n \geq 0\} \cup \{-\alpha + n\delta : \alpha \in R^+, n > 0\}$

$\cup \{n\delta : n > 0\}$.

$\hat{\mathfrak{n}}^\pm$

$\lambda \in \hat{\mathfrak{h}}^*$ $\quad M(\lambda) = U(\hat{\mathfrak{g}}) \Big/ \left\{\begin{matrix}\hat{\mathfrak{n}}^+ \\ h - \lambda(h) : h \in \hat{\mathfrak{h}}\end{matrix}\right\}$

$M(\lambda)$ has a ! irr. quot $V(\lambda)$

wt. mod $\quad$ wt $V(\lambda) \subseteq$ wt $M(\lambda) \subseteq \lambda - Q^+$.

$\hat{\mathcal{O}} \ni M$, $\quad M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$

dim $M_\mu < \infty$, $\quad$ wt $M \subseteq \bigcup_{s=1}^{k} \lambda_s - Q^+$

pause for a minute: - what is the most natural rep. of a Lie alg - adjoint

$\hat{g} \to \text{end}\, \hat{g}$ but $\hat{g} \notin \hat{\Theta}$

wts of $\hat{g}$ = $\hat{R} \cup \{0\} \subseteq \cancel{\bigcup_{s=1} \lambda_s} - Q^+$

$\Rightarrow n\delta \in \bigcup_{s=1} \lambda_s - Q^+$ but this is imposs.

$n > 0$

totally trivial to construct examples of reps of $\hat{g}$ which are not in $\hat{\Theta}$.

V any rep of $\underline{g}$, $V \in \Theta$

$V \otimes \mathbb{C}[t, t^{-1}] = L(V)$

$\hat{g}$ on $L(V)$ by $(x \otimes t^n)(v \otimes t^m)$

$= xv \otimes t^{n+m}$.

then $L(V) \notin \hat{\Theta}$. $\quad d\,(v \otimes t^n) = n v \otimes t^m$

$\underline{h(v \otimes t^n)} = \mu(h)$
μ

μ wt $\quad \cancel{\mu(v)}_{\mu} \quad L(V) = \bigoplus \bigoplus V_\mu \otimes t^n$

$\mu \in \mathfrak{h}^* \; n \in \mathbb{Z}$

$\mu + n\delta$ wt $\forall\, n > 0$.

something diff

analog of fd modules. - integrable

modules ~~ie for each~~

V integrable if V is a wt mod. for each $i \in 0, 1, \dots, n$,

and $v \in V$ $\exists$ $N = N(i, v)$ s.t.

$(x_{\alpha_i}^{\pm})^N \cdot v = 0$

Lemma: V integ wt mod.

then $V_\mu \neq 0 \Rightarrow \mu(c) \in \mathbb{Z}$.

Pf: $v \in V_\mu$. $\{x_{-\theta} \otimes t, x_\theta \otimes t^{-1}, c - h_\theta\}$

$C h \theta$ which is $\cong sl_2$

$\Rightarrow$ eigenvalue of $\mu(c - h_\theta) \in \mathbb{Z}$

similar $\mu(h_\theta) \in \mathbb{Z}$

V any $\hat{g}$-mod, V wt

$V = \oplus V^a$, $V^a = \bigoplus_{\substack{\mu \in h^* \\ \mu(c) = a}} V_\mu$

V^a $\hat{g}$-submod

V is integrable, say V has level k if c acts on V by k

<u>Thm</u>: (Kac) Let $V \in \hat{\mathcal{O}}$, V integrable. Then V is completely ~~irr~~educible i.e. is a direct sum of irr. modules

irr. integ modules are

$\{V(\lambda), : \lambda \in \hat{P}^+\}$ $\quad \lambda(h_i) \in \mathbb{Z}_+ \ \forall$
$i = 0, \ldots, n.$

$\dim V(\lambda) = \infty$ unless $\lambda = r\delta$ in which case $\dim V(r\delta) = 1$.

$V(\lambda)$, $\lambda \in \hat{P}^+$
positive level integ modules

$\lambda(c) = 1$ called a level one module.

$\lambda(h) = 0$, $h \in \mathfrak{h}$, $\lambda(c) = 1$ basic rep.

— vertex algebras / -- connections with number theory, monster gp etc.

$V(\lambda)$, $\lambda \in \hat{P}^+$ have other nice props

similar to fd reps of simple Lie alg

- recent ones: canonical basis
  global crystal basis.

- Weyl-Kac character formula exact
  same formula replace W by $\hat{W}$.

- gen and rel^n $V(\lambda)$

$$\hat{n}^+ V(\lambda) = 0 = (x_i^-)^{\lambda(h_i)+1} \cdot v_\lambda = 0$$

$$h v_\lambda = \lambda(h) v_\lambda .$$

what about negative level integ modules

$\lambda \in \hat{P}^-$ we never worry about this

for simple Lie alg $\exists$ $w_0 \in W$ longest element which does not exist in $\hat{W}$

how are the results on positive integrable modules proved?

$U(\hat{g})$ is not noetherian & no JHS

$Z(U(\hat{g})) \simeq \mathbb{C}[c]$ (Chari/Pressley?)

<u>Thm (Kac)</u> (i) $V \in \mathcal{O}$, $\lambda \in \mathfrak{h}^*$ $\exists$ filt.

$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_t = \{0\}$

s.t either $V_i / V_{i+1} \simeq V(\mu)$ for some $\mu > \lambda$

or $\left(V_i / V_{i+1} \right)_\mu = 0$.

(ii) Let $\lambda, \mu \in \mathfrak{h}^*$ $\mu > \lambda$ ~~then~~ ~~and let~~ then

$[V : V(\mu)]$ is the no. of times

$V(\mu)$ occurs in a filt. of the above type

<u>Center</u>: Ω. Casimir element.

pick a basis of $\mathfrak{g}$ x_i

dual basis w.r.t Killing form x^i

$\Omega = \sum x_i x^i$

$\hat{g}$ – no killing form

– symmetric non-deg bilinear form.
– can be defined.

$\Omega = \sum x_i x^i$ infinite sum

$$= \sum_{\substack{n>0 \\ \alpha \in R}} x_{-\alpha} \otimes t^{-n} x_\alpha \otimes t^n + \sum_{i, n>0} h_i \otimes t^{-n} h_i \otimes t^n.$$

$\forall\, v \in \mathcal{O}$ $(x_\alpha \otimes t^n) v = 0 \;\forall n >> 0$

$\Rightarrow$ Ω acts on V and commute with $\hat{g}$ action and this is enough!

Deodhar - Kac - Gabber blocks for $\hat{\mathcal{O}}$

$\mathcal{O} = \oplus \mathcal{O}_\chi$ $\hat{\mathcal{O}} = \oplus \mathcal{O}_\chi$

Peter Fiebig – generalize work of Soergel etc.

– level zero reps –