≪ Quantum Groups & Crystal Bases I ≫

6/15/09

Ottawa

1. Lie algs & Hopf algs

G : alg group = alg variety + group

Lie group = manifold + group

$T_x(G)$ = tangent space of G at $x \in G$

$\mathbb{F}[G]$ = alg of regular fns on G

= " ~~differentiable~~ smooth fns on G

$D_x(G) = \{ d : \mathbb{F}[G] \to \mathbb{F} \mid$ linear mapping s.t. $d(fg) = d(f)g(x) + f(x)d(g) \}$, $x \in G$

$$\Rightarrow T_x(G) \xrightarrow{\sim} D_x(G)$$

$$v \longmapsto d_v : f \mapsto Df(x) \cdot v$$

$$X \longmapsto d_X : f \mapsto Df(x) \cdot X$$

$\mathcal{L}(G) = \{$ left invariant derivations of $\mathbb{F}[G] \}$

$= \{ D : \mathbb{F}[G] \to \mathbb{F}[G] \mid$ linear map,

$D(fg) = D(f)g + fD(g),\ D\lambda(x) = \lambda(x)D\ \forall x \in G \}$,

where $(\lambda(x)f)(y) = f(xy)\ \forall x, y \in G$.

Recall

$\Rightarrow T_1(G) \xrightarrow{\sim} D_1(G) \cancel{\xrightarrow{\sim} \mathcal{L}(G)}$

$X \longmapsto d_X \cancel{\longmapsto D_X}$,

Def: $D_X : \mathbb{F}[G] \to \mathbb{F}[G]$ by

$f \longmapsto D_X(f) : x \longmapsto d_X(\lambda(x)f)\quad x \in G$.

(Exercise) $D \in \mathcal{L}(G) \Rightarrow D = D_X$ for a unique $X \in G$

$(\because)$ Define $d \in D_1(G)$ by $d(f) = D(f)(1)$.

$\Rightarrow d = d_X$ for a unique $X \in T_1(G)$

Exercise $\boxed{\text{Prop}}$ $\boxed{T_1(G) \xrightarrow{\sim} \mathcal{L}(G)}$

<pf> (Exercise)

Define $[D_X, D_Y] = D_X D_Y - D_Y D_X$.

$\Rightarrow [D_X, D_Y] \in \mathcal{L}(G)$; i.e. $[D_X, D_Y] = D_Z$, $Z \in T_1(G)$

Thus we may write $[D_X, D_Y] = D_{[X,Y]}$, $Z = [X,Y]$.

Moreover, $\mathcal{L}(G)$ becomes a <u>Lie alg</u> ; i.e.

$$[D_X, D_X] = 0$$

$$[D_X, [D_Y, D_Z]] + [D_Y, [D_Z, D_X]] + [D_Z, [D_X, D_Y]] = 0.$$

Def $\mathfrak{g} = \mathrm{Lie}(G) = \mathcal{L}(G) \cong T_1(G)$

$=$ the <u>Lie algebra</u> of G.

Def L : Lie alg / $\mathbb{F}$

$=$ vector space / $\mathbb{F}$ with a bilinear product

$[\ ,\ ] : L \times L \to L$ satisfying

i) $[x, x] = 0 \quad \forall x \in L$

ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$

(Example) ① $L = \mathbb{R}^3$, $[x,y] = x \times y$, cross product

② A : assoc. alg / $\mathbb{F}$

$$[x,y] = xy - yx \quad \forall x,y \in A.$$

$\Longrightarrow L = (A, [\,,\,])$: Lie alg

③ V : vector sp / $\mathbb{F}$

$L = (A = \mathrm{End}_{\mathbb{F}} V, [\,,\,]) = gl(V)$

= <u>general</u> <u>linear</u> <u>Lie</u> alg of V

④ $V = \mathbb{F}^n$; $gl(V) = gl(n, \mathbb{F}) = gl_n(\mathbb{F})$.

⑤ $L = sl(n, \mathbb{F}) = \{x \in gl(n, \mathbb{F}) \mid \mathrm{tr}\, x = 0\}$

= <u>special</u> <u>linear</u> <u>Lie</u> alg

$\boxed{\text{Def}}$ ① Lie subalg ② ideal

③ homomorphism ④ simple Lie alg

$\boxed{\text{Prob}}$ How to classify (f.d.) simple Lie alg? Cartan

Def L : Lie alg / $\mathbb{F}$

V : vector space / $\mathbb{F}$

① A <u>representation</u> of L on V

= a Lie alg homomorph. $\phi : L \to \mathfrak{gl}(V)$

② V is an L-module if $\exists$ a bilinear map

$L \times V \to V$, $(x, v) \mapsto x \cdot v$ s.t

$[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ $\forall x, y \in L$, $\forall v \in V$.

Rmk ① $\Leftrightarrow$ ② ; $x \cdot v = \phi(x)(v)$

$\triangle$ repn theory = study of L-modules

Def ① <u>subrepn</u> or submodule

$W \leq V$ if $\phi(L)(W) \subset W$;

i.e. $x \cdot W \subset W$ $\forall x \in L$.

② irreducible repns

~~Given a Lie alg L,~~

Prob How to construct & classify irreducible L-modules ?

$\boxed{\text{Def}}$ L: Lie alg $/\mathbb{F}$, U: assoc. alg $/\mathbb{F}$

$\iota : L \to U(L)$ linear map satisfying

$$\iota([x,y]) = \iota(x)\iota(y) - \iota(y)\iota(x) \quad \forall x, y \in L.$$

The pair (U, ι) is a universal enveloping alg of L if $\forall$ pair (A, j), A: alg, $j : L \to A$ satisfying $(*)$, $\exists!$ alg homo $\phi : U \to A$ s.t. $\phi \circ \iota = j$;

$$\begin{tikzcd} L \arrow[r, "\iota"] \arrow[dr, "j"'] & U \arrow[d, dashed, "\exists!\phi"] \\ & A \end{tikzcd}$$

(Uniqueness) Usual argument.

(Existence) $U = T(L)/I$, $T(L) = \bigoplus_{k \geq 0} L^{\otimes k}$,

$$I = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in L \rangle.$$

$\rightsquigarrow$ The universal enveloping alg of L

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~~Prop~~ $\boxed{\text{Thm}}$ (PBW Thm)

① $\iota : L \to T(L) \to U(L)$ is injective

② $\{x_\alpha \mid \alpha \in \Omega\}$: ordered basis of L

$\Rightarrow \{x_{i_1} x_{i_2} \cdots x_{i_r} \mid r \geq 0,\ i_1 \leq \cdots \leq i_r,\ i_k \in \Omega\}$

forms a basis of $U(L)$.

$\boxed{\text{Rmk}}$ ① V is a L-module

$\Leftrightarrow V$ is a $U(L)$-module

~~② Study rep. theory = study of L-modules~~

Question; ① V, W : L-modules $\overset{?}{\Rightarrow} V \otimes W$ is an L-module?

② V : L-module $\overset{?}{\Rightarrow} V^*$: L-module.

Yes; $\left(\begin{array}{l} x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w \\ (x \cdot f)(v) = -f(x \cdot v) \end{array} \right.$

Actually, this is possible because $U(L)$ is a Hopf alg; i.e. $\exists$ alg hom $\Delta : U(L) \to U(L) \otimes U(L)$ and an anti-involution $S : U(L) \to U(L)$ defined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x \quad \forall x \in L.$$

$\boxed{\text{Def}}$ A : assoc alg $/\mathbb{F}$

= vector sp $/\mathbb{F}$ with linear maps

$\mu : A \otimes A \to \mathbb{F}$, $\iota : \mathbb{F} \to A$ s.t.

$$\begin{tikzcd} A \otimes \mathbb{F} \arrow[r, "\mathrm{id} \otimes \iota"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \otimes A \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

~~Def $\phi : A \to A'$ is a hom of algebras if~~

$\phi \circ \mu_A = \mu_{A'}$,

$\boxed{\text{Def}}$ $\phi : A \to A'$ is an alg homomorphism

if $\phi \circ \iota_A = \iota_{A'}$, $\phi \circ \mu_A = \mu_{A'} \circ (\phi \otimes \phi)$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "\iota_A"] \arrow[dr, "\iota_{A'}"'] & A \arrow[d, "\phi"] \\ & A' \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "\phi\otimes\phi"'] & A \arrow[d, "\phi"] \\ A' \otimes A' \arrow[r, "\mu_{A'}"'] & A' \end{tikzcd}$$

$\boxed{\text{Def}}$ A: coalg / $\mathbb{F}$

= vect sp / $\mathbb{F}$ with linear maps

$\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{F}$ s.t.

$$\begin{tikzcd} A \otimes \mathbb{F} & A \otimes A \arrow[l, "id \otimes \varepsilon"'] \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes id"'] \\ A \otimes A \arrow[u, "id \otimes \Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

Def $\phi : A' \to A$ is a <u>coalg</u> <u>homomorphism</u>

if $\varepsilon_A \circ \phi = \varepsilon_{A'}$, $(\phi \otimes \phi) \circ \Delta_{A'} = \Delta_A \circ \phi$

$$\begin{tikzcd}
\mathbb{F} & A \arrow[l, "\varepsilon_A"'] & & A \otimes A & A \arrow[l, "\Delta_A"'] \\
& A' \arrow[ul, "\varepsilon_{A'}"] \arrow[u, "\phi"'] & & A' \otimes A' \arrow[u, "\phi \otimes \phi"] & A' \arrow[l, "\Delta_{A'}"] \arrow[u, "\phi"']
\end{tikzcd}$$

Def $\phi : A \to A'$ is a <u>coalg</u> <u>homomorphism</u>

if $\varepsilon_{A'} \circ \phi = \varepsilon_A$, $(\phi \otimes \phi) \cdot \Delta_A = \Delta_{A'} \circ \phi$

$$\begin{tikzcd}
\mathbb{F} & A \arrow[l, "\varepsilon_A"'] \arrow[dl, phantom, "\sim"] \arrow[d, "\phi"] & & A \otimes A \arrow[d, "\phi \otimes \phi"] & A \arrow[l, "\Delta_A"'] \arrow[d, "\phi"] \\
& A' \arrow[ul, "\varepsilon_{A'}"] & & A' \otimes A' & A' \arrow[l, "\Delta_{A'}"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "\varepsilon_A"] \arrow[dr, "\phi"'] & & \mathbb{F} & & A \arrow[r, "\Delta_A"] \arrow[d, "\phi"'] & A \otimes A \arrow[d, "\phi \otimes \phi"] \\
& A' \arrow[ur, "\varepsilon_{A'}"'] & & & A' \arrow[r, "\Delta_{A'}"'] & A' \otimes A'
\end{tikzcd}$$

$\boxed{\text{Def}}$ A : Hopf alg / $\mathbb{F}$

= vector sp / $\mathbb{F}$ together with linear maps

$\mu: A\otimes A \to \mathbb{F}$, $\Delta: A \to A\otimes A$

$\iota: \mathbb{F} \to A$, $\varepsilon: A \to \mathbb{F}$, $S: A \to A$

s.t. i) (A, μ, ι) : alg

ii) (A, Δ, ε) : coalg

iii) ~~(μ, ι) :~~ μ & ι : coalg homomorph

iv) Δ, ε : alg homomorph

v) $\mu\circ(S\otimes id)\circ\Delta = \iota\circ\varepsilon$, $\mu\circ(id\otimes S)\circ\Delta = \iota\circ\varepsilon$

$$\begin{tikzcd} A\otimes A \arrow[r, "S\otimes id"] & A\otimes A \arrow[d, "\mu"] \\ A \arrow[u, "\Delta"] \arrow[r, "\iota\circ\varepsilon"'] & A \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[r, "id\otimes S"] & A\otimes A \arrow[d, "\mu"] \\ A \arrow[u, "\Delta"] \arrow[r, "\iota\circ\varepsilon"'] & A \end{tikzcd}$$

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$S(x^{\pm 1}) = x^{\mp 1}$

(Example) ⓪ $A = \mathbb{F}[x^{\pm 1}]$, $\Delta(x^{\pm 1}) = x^{\pm 1} \otimes x^{\pm 1}$, $\varepsilon(x^{\pm 1}) = 1$,

① G : group, $A = \mathbb{F}G$: group alg

$\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$

noncomm, cocomm Hopf alg

② $A = U(L)$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$

noncomm. cocomm Hopf alg

③ G : alg group, $A = \mathbb{F}[G] = \{\text{fncts from } G\}$

$\Delta(f)(x, y) = f(xy)$, $\varepsilon(f) = 1$, $S(f)(x) = f(x^{-1})$,

Δ: $\mathbb{F}[G] \otimes \mathbb{F}[G] \hookrightarrow \mathbb{F}[G \times G]$.

($\# G < \infty \Rightarrow \mathbb{F}[G] \otimes \mathbb{F}[G] \simeq \mathbb{F}[G \times G]$).