



« Quantum Groups & Crystal Bases I »

6/15/09

L. Lie algs & Hopf algs

Ottawa

G : alg group = alg variety + group

L group = manifold + group

$T_x(G)$ = tangent space of G at $x \in G$

$F[G]$ = alg of regular fns on G

= " ~~differentiable~~ smooth fns on G

$D_x(G) = \{d: F[G] \rightarrow F \mid \text{linear mapping}\}$
s.t. $d(fg) = d(f)g(x) + f(x)d(g)$, $x \in G$

$$\Rightarrow T_x(G) \xrightarrow{\sim} D_x(G)$$

$$v \longmapsto d_v: f \longmapsto Df(x) \cdot v$$

$$X \longmapsto d_x: f \longmapsto Df(x) \cdot X$$



$$L(G) = \{ \text{left invariant derivs of } \mathbb{F}[G] \}$$

$$= \{ D: \mathbb{F}[G] \rightarrow \mathbb{F}[G] \mid \text{linea map,}$$

$$D(fg) = D(f)g + fD(g), D(\lambda(x)) = \lambda(x)D \forall x \in G, \}$$

where $(\lambda(x)f)(y) = f(xy) \forall x, y \in G.$

Recall

$$\begin{array}{ccc} T_1(G) & \xrightarrow{\sim} & D_1(G) & \xrightarrow{\sim} & L(G) \\ \downarrow \psi & & \downarrow & & \downarrow \\ X & \xrightarrow{1} & d_x & \xrightarrow{1} & D_x \end{array}$$

Def: $D_x: \mathbb{F}[G] \rightarrow \mathbb{F}[G]$ by

$$f \mapsto D_x(f): g \mapsto d_x(\lambda(g)f) \quad x \in G.$$

(Exercise) $D \in L(G) \Rightarrow D = D_x$ for a unique $x \in G.$

(\because) $(D \text{ is } d \in D_1(G) \text{ by } d(f) = D(f)(1).$

$\Rightarrow d = d_x$ for a unique $x \in T_1(G)$

(Exercise) Prop $T_1(G) \xrightarrow{\sim} L(G)$

$\langle \text{ps} \rangle$ (Exercise)



Define $[D_x, D_y] = D_x D_y - D_y D_x$.

$\Rightarrow [D_x, D_y] \in \mathfrak{L}(G)$; i.e. $[D_x, D_y] = D_z, z \in \mathfrak{T}_1(G)$

Thus we may write $[D_x, D_y] = D_{(x,y)}, z = (x,y)$.

Moreover, $\mathfrak{L}(G)$ becomes a Lie algebra ; i.e.

$$[D_x, D_x] = 0$$

$$[D_x, [D_y, D_z]] + [D_y, [D_z, D_x]] + [D_z, [D_x, D_y]] = 0.$$

Def $\mathfrak{g} = \text{Lie}(G) = \mathfrak{L}(G) \cong \mathfrak{T}_1(G)$
= the Lie algebra of G .

Def $L = \text{Lie } \mathfrak{g} / \mathbb{F}$
= vector space \mathbb{F} with a bilinear product
 $[,] : L \times L \rightarrow L$ satisfies

i) $[x, x] = 0 \quad \forall x \in L$

ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$



(Example) ① $L = \mathbb{R}^3$, $[x, y] = x \times y$, cross product

② A : assoc. alg of \mathbb{F}

$$[x, y] = xy - yx \quad \forall x, y \in A$$

$\Rightarrow L = (A, [,]) : \text{Lie alg}$

③ V : vector sp of \mathbb{F}

$$\mathbb{R} \cong (A = \text{End}_{\mathbb{F}} V, [,]) = \mathfrak{gl}(V)$$

= general linear Lie alg of V

$$\textcircled{4} V = \mathbb{F}^n : \mathfrak{gl}(V) = \mathfrak{gl}(n, \mathbb{F}) = \mathfrak{gl}_n(\mathbb{F})$$

$$\textcircled{5} L = \mathfrak{sl}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, \mathbb{F}) \mid \text{tr} x = 0\}$$

= special linear Lie alg

Def

① Lie subalg

② ideal

③ homomorphism

④ simple Lie alg

Prob

How to identify (f.d.) simple Lie algs?
(Contd 2)



Def

L : Lie alg / F
 V : vector space / F

① A representation of L on V
 = a Lie alg homomorphism $\phi: L \rightarrow \mathfrak{gl}(V)$

② V is an L -module $\iff \exists$ a bilinear map
 $L \times V \rightarrow V, (x, v) \mapsto x \cdot v$ s.t

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \forall x, y \in L, \forall v \in V.$$

Rule

① \iff ② : $x \cdot v = \phi(x)(v)$

~~⊗~~ rep theory
 = study of L -modules

Def

① sub rep or submodule

$W \subseteq V \iff \phi(L)(W) \subset W$;

i.e. $x \cdot W \subset W \quad \forall x \in L.$

② irreducible reps

Prob ~~Give a list of~~
 How to construct & classify module

L -modules?



Def

L : k -alg of $/F$, U : k -space of $/F$

$\iota: L \rightarrow U(U)$ ~~is a map~~

$$\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x) \quad \forall x, y \in L.$$

The pair (U, ι) is a universal enveloping algebra of L

if \forall pair (A, j) , $A = U$ - k -alg $A = L \rightarrow A$

exists $(\exists!)$ k -homo $\phi: U \rightarrow A$ s.t.

$$\phi \circ \iota = j; \quad \begin{array}{ccc} L & \xrightarrow{\iota} & U \\ & \searrow j & \swarrow \exists! \phi \\ & & A \end{array}$$

(Uniqueness) Universal algebra.

(Existence) $U \cong T(U)/I$, $T(U) = \bigoplus_{k \geq 0} L^{\otimes k}$,

$$I = \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in L \rangle.$$

\leadsto The universal enveloping algebra of L



PBW T_{un} (PBW T_{un})

- ① $\epsilon: L \rightarrow T(L) \rightarrow U(L)$ is injective
- ② $\{\alpha_i \mid i \in \Omega\}$: ordered basis of L
- $\Rightarrow \{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_r} \mid r \geq 0, i_1 \leq \dots \leq i_r, i_k \in \Omega\}$ forms a basis of $U(L)$.

R_{un} $\otimes V$ is an L -module

$\Leftrightarrow V$ is a $U(L)$ -module

~~② Study $\text{Hom}(V, V) = \text{study of } L\text{-modules}$~~

Question: ① $V, W: L\text{-modules} \Rightarrow V \otimes W$ is an L -module?

② $V: L\text{-module} \Rightarrow \mathbb{Q}V^*: L\text{-module}$.

Yes; $(x \cdot (v \otimes w)) = x \cdot v \otimes w + v \otimes x \cdot w$
 $(x \cdot f)(w) = -f(x \cdot v)$



Actually, this is possible because $D(L)$ is a Hopf algebra. \exists alg hom $\Delta: D(L) \rightarrow D(L) \otimes D(L)$ and an anti-involution $S: D(L) \rightarrow D(L)$ defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x \quad \forall x \in L$.

Def

A : assoc alg / \mathbb{F}

= vector sp / \mathbb{F} with brackets

$\mu: A \otimes A \rightarrow A$, $\iota: \mathbb{F} \rightarrow A$ s.t

$$\begin{array}{ccc} A \otimes \mathbb{F} & \xrightarrow{\text{id} \otimes \iota} & A \otimes A \\ \downarrow \cong & & \downarrow \mu \\ & & A \end{array}$$

$$\begin{array}{ccc} \mathbb{F} \otimes A & \xrightarrow{\iota \otimes \text{id}} & A \otimes A \\ \downarrow \cong & & \downarrow \mu \\ & & A \end{array}$$

$$A \otimes A \otimes A \xrightarrow{\text{id} \otimes \mu} A \otimes A$$

$$\begin{array}{ccc} \text{id} \otimes \mu & \searrow & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

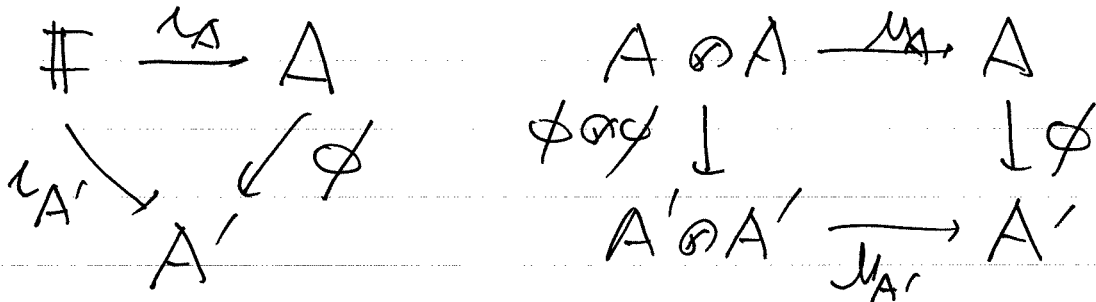
~~**Def** $\phi: A \rightarrow A'$ is a map of algebras~~

$$\phi \circ \mu_A = \mu_{A'}$$



Def $\phi: A \rightarrow A'$ is an algebra homomorphism

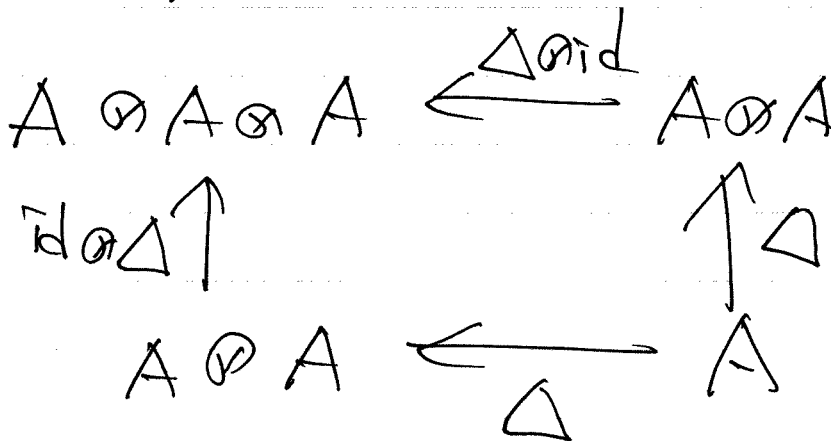
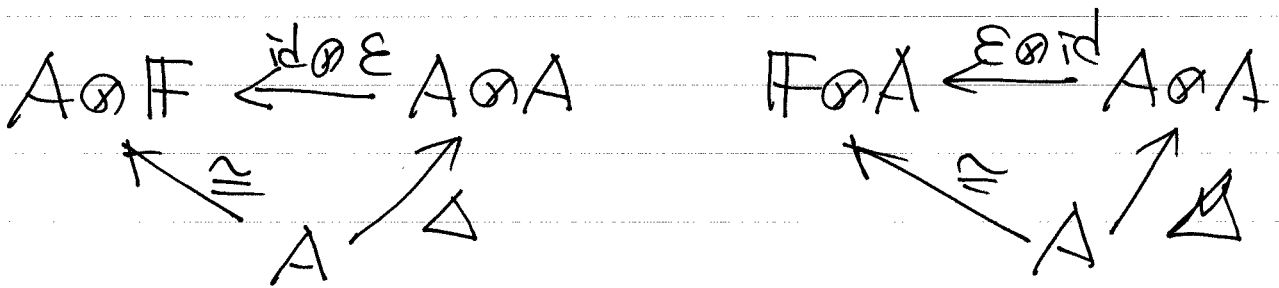
$$\# \phi \circ \iota_A = \iota_{A'}, \quad \phi \circ \mu_A = \mu_{A'} \circ (\phi \otimes \phi)$$



Def $A: \text{alg} / \mathbb{F}$

= vect sp / \mathbb{F} with linear maps

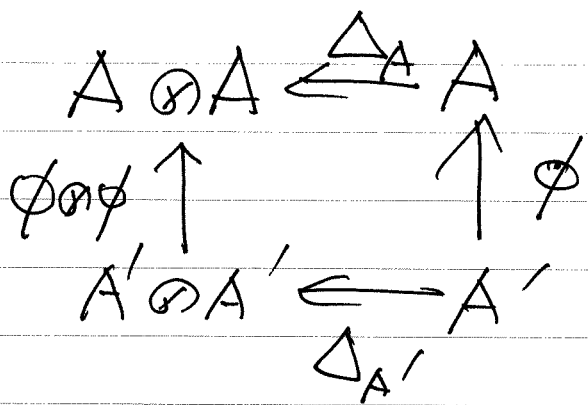
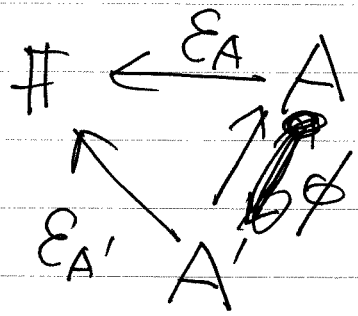
$$\Delta: A \rightarrow A \otimes A, \quad \varepsilon: A \rightarrow \mathbb{F} \text{ s.t.}$$





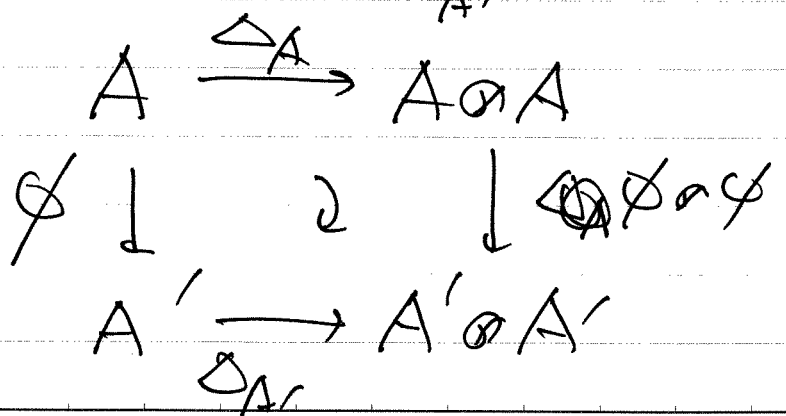
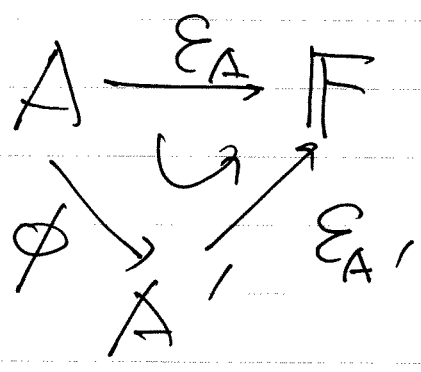
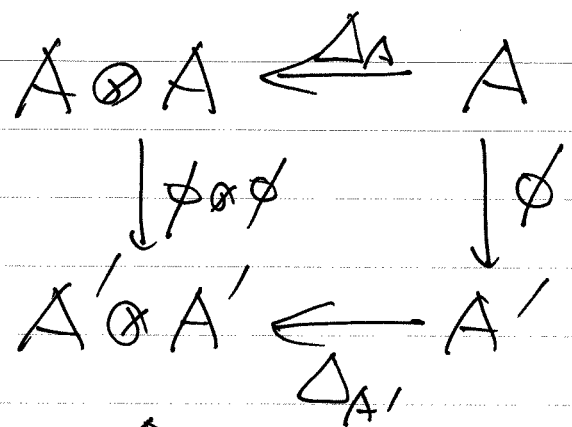
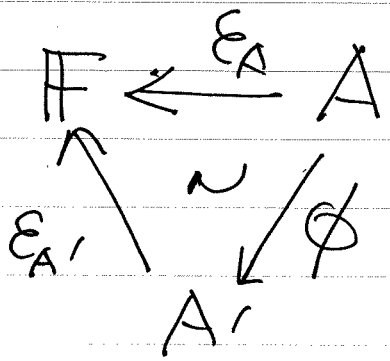
D_{co} $\phi: A \rightarrow A'$ is a coalgebra

$\varepsilon_A \circ \phi = \varepsilon_{A'}$, $(\phi \circ \phi) \circ \Delta_{A'} = \Delta_A \circ \phi$



D_{co} $\phi: A \rightarrow A'$ is a coalgebra

$\varepsilon_{A'} \circ \phi = \varepsilon_A$, $(\phi \circ \phi) \circ \Delta_A = \Delta_{A'} \circ \phi$





Def

$$A : \text{Hopf alg} / \mathbb{F}$$

= vector sp / \mathbb{F} together with linear maps

$$\mu : A \otimes A \rightarrow A, \quad \Delta : A \rightarrow A \otimes A$$

$$\iota : \mathbb{F} \rightarrow A, \quad \varepsilon : A \rightarrow \mathbb{F}, \quad S : A \rightarrow A$$

s.t. i) $(A, \mu, \iota) : \text{alg}$

ii) $(A, \Delta, \varepsilon) : \text{coalg}$

iii) ~~(μ, ι)~~ : μ & ι : coalg homomorph

iv) Δ, ε : alg homomorph

~~v) S is an involution~~

v) $\mu_0(S \circ \text{id}) \circ \Delta = \iota \circ \varepsilon, \quad \mu_0(\text{id} \circ S) \circ \Delta = \iota \circ \varepsilon$

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{S \circ \text{id}} & A \otimes A \\
 \Delta \uparrow & & \downarrow \mu \\
 A & \xrightarrow{\iota \circ \varepsilon} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \otimes A & \xrightarrow{\text{id} \circ S} & A \otimes A \\
 \Delta \uparrow & & \downarrow \mu \\
 A & \xrightarrow{\iota \circ \varepsilon} & A
 \end{array}$$



$$S(x) = x^{-1}$$

①

$$A = \mathbb{F}(x)$$

$$\Delta(x) = x \otimes x^{-1}$$

$$\varepsilon(x) = 1$$

$$S(x) = x^{-1}$$

②

(Example)

①

G : group, $A = \mathbb{F}G$: group alg

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

noncomm, cocomm Hyl alg

②

$$A = \mathbb{O}(L)$$

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$S(x) = -x$$

noncomm. comm Hyl alg

③

G : alg group, $A = \mathbb{F}[G]$ - free from G

$$\Delta(f)(x, y) = f(xy), \quad \varepsilon(f) = 1, \quad S(f)(x) = f(x^{-1})$$

$$\text{free } \mathbb{F}\langle G \rangle \otimes \mathbb{F}\langle G \rangle \longleftrightarrow \mathbb{F}\langle G \times G \rangle$$

$$(\text{if } G \text{ is } \infty \Rightarrow \mathbb{F}\langle G \rangle \otimes \mathbb{F}\langle G \rangle \cong \mathbb{F}\langle G \times G \rangle)$$