Risk, Return, and Ross Recovery

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Overview of this Talk

- There are five parts to this talk:

  1. Review of Fundamental Theorems of Asset Pricing
  2. Ross Recovery Theorem for Finite State Markov Chains
  3. Review of John Long's Numeraire Portfolio
  4. Preference-Free Ross Recovery for Bounded Diffusions
  5. First Results for Unbounded Diffusions

- The operating assumptions will be different in each section. Within a section, only one set of assumptions holds.

Part I: Fundamental Theorems of Asset Pricing

- The term "Fundamental Theorems of Asset Pricing" (FTAP) was coined by Phil Dybvig, to describe work initiated by his thesis advisor, Steve Ross, around 1978.
- The connection to martingale theory is in Kreps (1979), and in Harrison & Kreps (1979), with an important extension to the continuous time setting in Harrison and Pliska (1981).
- The most general version of FTAP is by Delbaen and Schachermayer, who now have a book on the subject.
- There are actually 2 FTAP's, one that describes the implications of no arbitrage, and a second one that further supposes that markets are complete.

First FTAP

- The "First Fundamental Theorem of Asset Pricing" relates two types of probability measures usually denoted $\mathbb{P}$ and $\mathbb{Q}$.
- In this talk, the probability measure $\mathbb{P}$ indicates the frequency with which "the market" believes that future states occur. We assume that this frequency is reflected in market prices, which also reflect investor attitudes towards risk. $\mathbb{P}$ need not be "true" or "real world" probability.
- Loosely, a market is said to be "arbitrage-free" if there is no way to form a portfolio which can't lose and might win (under $\mathbb{P}$).
- Mathematically, FTAP#1 assumes $\mathbb{P}$ exists along with an arbitrage-free market containing a money market account with price $S_{0t} > 0, t \geq 0$. FTAP#1 says that these assumptions imply the existence of another probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for each asset's spot price $S_{it} \in \mathbb{R}, i = 0, 1 \ldots, n$, the relative price $\frac{S_{it}}{S_{0t}}$ is a $\mathbb{Q}$ martingale.
- The probability measure $\mathbb{Q}$ is referred to as an "equivalent martingale measure" (EMM) and is also called a "risk-neutral measure".

Second FTAP

- Loosely, a financial market is said to be "complete" if the payoff of any contingent claim in a specified set can be replicated by dynamic trading in a specified set of so-called "basis assets".
- FTAP#2 says that when FTAP#1 holds, a market is complete if and only if the EMM $\mathbb{Q}$ is unique.
- In this talk, we will only consider arbitrage-free markets and then impose whatever assumptions we need to e.g. completeness, so that $\mathbb{Q}$ is unique.
- We will know the unique $\mathbb{Q}$, but not the probability measure $\mathbb{P}$ from which it supposedly sprang. We will impose restrictions such that the measure change $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is unique and becomes known. As a result the probability measure $\mathbb{P}$ is unique and we will learn $\mathbb{P}$ from $\mathbb{Q}$ and our restrictions. We say that we have recovered $\mathbb{P}$ from $\mathbb{Q}$ (and our assumptions).

Change of Numeraire

- A numeraire is a self-financing portfolio whose value is always positive.
- As presented thus far, the two FTAP's use the money market account (MMA) as numeraire.
- Merton 73 and Margrabe 78 consider alternative numeraires when pricing options. The FTAP's were extended to alternative numeraires by Géman, El Karoui, and Rochet (1995).
- FTAP#1 says that no arbitrage between assets with spot prices $S_i, i = 0, 1, \ldots$ implies that for each numeraire with spot price $N_n > 0, n = 0, 1, \ldots$, there exists a probability measure $\mathbb{Q}^n$ equivalent to $\mathbb{P}$ such that $\frac{S_i}{N_n}, i = 0, 1, \ldots$ is a $\mathbb{Q}^n$ martingale for $n = 0, 1, \ldots$.
- For example, if $S_0 > 0$ is the spot price of the MMA and if we use this MMA as the first numeraire, i.e. $N_0 = S_0$, then no arbitrage implies there exists $\mathbb{Q}^0$ equivalent to $\mathbb{P}$ such that $\frac{S_i}{S_0}, i = 0, 1, \ldots$ is a $\mathbb{Q}^0$ martingale.
- In general, changing the numeraire while fixing the probability measure changes the drift of a relative price. For example, with the probability measure fixed at $\mathbb{Q}^0$, the relative price S_i/S_0 is a $\mathbb{Q}^0$ martingale, but the

Part II: $\mathbb{P}$, $\mathbb{Q}$, and Ross Recovery

- Recall that the probability measure $\mathbb{P}$ captures the extent to which the market's belief about the frequencies of future states ends up in market prices. Suppose that $\mathbb{P}$ is ex ante unknown by us, but we know market prices.
- From prices and a sufficiently strong set of assumptions, we can learn the risk-neutral probability measure $\mathbb{Q}$. Having done so, we know $\mathbb{Q}$ ex ante but not $\mathbb{P}$.
- In 2011, Steve Ross began circulating a working paper called "The Recovery Theorem" whose first theorem gives sufficient conditions under which knowing $\mathbb{Q}$ implies knowing $\mathbb{P}$ exactly.
- We call his Theorem 1 the Ross Recovery Theorem.

The Ross Recovery Theorem

- Theorem 1 in Ross (2011) states that:
    1. if markets are complete, and
    2. if the utility function of the representative investor is state-independent and intertemporally additively separable with a constant rate of time preference and:
    3. if there is a single state variable X which under $\mathbb{Q}$ is a time-homogeneous Markov chain with a finite number of states,
- then under $\mathbb{P}$, X is also a finite-state time-homogeneous Markov chain and we can recover the transition probability matrix P of X from the assumed known risk-neutral transition probability matrix Q.

Vas Ist Das?

- A couple of us at Morgan Stanley were intrigued by Ross's conclusion that $\mathbb{P}$ can be learned from knowledge of $\mathbb{Q}$ in a Markovian setting, but we wondered whether it was necessary to restrict preferences. Is there a preference-free way to obtain $\mathbb{P}$ from $\mathbb{Q}$ in a Markovian setting?
- We also wondered whether it was necessary that the Markovian state variable X transition between a finite number of states. The industry practice is to use models with a continuous state space. We wondered in particular if $\mathbb{P}$ can be learned from knowledge of $\mathbb{Q}$ when X is a diffusion under $\mathbb{Q}$.
- Finally, we wondered whether it was necessary that the state variable X drive the price of every asset in the whole economy. Ross's use of a representative investor requires that X drive the price of every asset in the economy. While some pairs of assets are highly correlated, many are not. Could we restrict ourselves to some strict subset of the economy where it was reasonable to believe that all prices are driven by a single state variable X?
- Could there be more than one way to recover $\mathbb{P}$ from $\mathbb{Q}$?

Preference-Free Ross Recovery

- We start by adjoining to the money market account a set of n risky assets, which in general would be a strict subset of all of the assets in the economy. In choosing this set, we require that it be reasonable to assume that:
  1. there is no arbitrage between the $n+1$ assets.
  2. a single state variable X drives all $n+1$ prices (under $\mathbb{Q}$).
- In the next section, we explore something called "the numeraire portfolio" which is composed out of just the $n+1$ assets.
- By restricting the $\mathbb{Q}$ dynamics of this numeraire portfolio, we show that there is a preference-free way to learn $\mathbb{P}$ from $\mathbb{Q}$ when the state variable X is a time homogeneous diffusion on a bounded domain.
- We also find an example where one can't learn $\mathbb{P}$ from $\mathbb{Q}$ when X diffuses over an unbounded domain. However, we find a second example of an unbounded diffusion where one can learn $\mathbb{P}$ from $\mathbb{Q}$. Knowing when you can and cannot recover on unbounded domains is at present an open problem.
- Under our current results, the interest rate and the prices of the assets themselves are allowed to be unbounded. It is only the driving process X which must be bounded.

Part III: John Long's Numeraire Portfolio

- In 1990, Long introduced a notion which he called the *numeraire portfolio*.
- Recall that a numeraire is any self-financing portfolio whose price is alway positive.
- Long showed that if any set of assets includes the MMA and is arbitrage-free, then there always exists a self-financing portfolio of them whose value is always positive.
- Furthermore, if the spot price S_i of each asset is expressed relative to the value $L > 0$ of Long's numeraire portfolio, then the relative price S_i/L is a $\mathbb{P}$ martingale.
- Long's discovery of the existence of the numeraire portfolio allows one to do arbitrage-free pricing of derivative securities without having to change measure away from $\mathbb{P}$. Instead, one deflates every asset's spot price S_i by the value L of Long's numeraire portfolio, and then uses the probability measure $\mathbb{P}$ as the martingale measure. This pricing approach would require knowing or learning both $\mathbb{P}$ and L.

Existence of the Numeraire Portfolio

- Let S_{0t} be the spot price of the MMA and suppose that we have n risky assets with spot prices S_i for $i = 1, \ldots, n$.
- Assuming no arbitrage between these $n + 1$ assets, Long (1990) proved that there exists a portfolio with value $L > 0$ such that for all times u and t with $u \geq t \geq 0$:
$$E^{\mathbb{P}} \frac{S_{iu}}{L_u}\bigg|_{\mathcal{F}_t} = \frac{S_{it}}{L_t}, \qquad i = 0, 1, \ldots, n.$$
- In words, assuming no arbitrage between a set of assets, implies that one can always construct a portfolio of them with value $L > 0$ such that each asset's relative price S_i/L is a $\mathbb{P}$ martingale. Hence, when P&L is measured in units of the numeraire portfolio, all assets have the same mean $P\&L$.

Intuition on the Properties of the Numeraire Portfolio

- Since the MMA is included in the list of $n+1$ assets making up the numeraire portfolio, it is reasonable to believe that one can construct a portfolio which always has positive value.
- However, most people find it difficult to believe that in an arbitrage-free economy, the risk premium on every asset can be removed merely by expressing the gains on each asset in units of the numeraire portfolio.
- Let's take a closer look at how this happens.

The Numeraire Portfolio in Discrete Time

- Suppose for simplicity that:
  1. trading is discrete in time
  2. the interest rate is always zero, and
  3. the spot price of each asset is strictly positive $S_i > 0, i = 0, 1, \ldots n$.
- Suppose an investor buys one share of asset i at time n at its spot price $S_{i,n}$.
- The total cost of this share purchase is $S_{i,n}$ dollars. Suppose the investor finances this upfront cost by borrowing $S_{i,n}$ dollars at time n.
- At time $n + 1$ the investor sells his share and repays his loan, leaving the real amount $S_{i,n+1} - S_{i,n}$ as the realized gain on the position.
- This gain can be invested in the MMA and can be carried forward unchanged to any later date $n + 2, n + 3$ etc.

The Numeraire Portfolio in Discrete Time (Con'd)

- Now suppose that our investor instead buys $\frac{1}{S_{i,n}}$ shares of asset i at time n with each share purchased at its spot price $S_{i,n}$.
- The total cost of this share purchase is one dollar. Suppose the investor finances this upfront cost by borrowing one dollar at time n.
- At time $n+1$ the investor sells the $\frac{1}{S_{i,n}}$ shares and repays the loan, leaving the real amount $\frac{S_{i,n+1}}{S_{i,n}} - 1$ as the realized gain on the financed position.
- Since $\frac{S_{i,n+1}}{S_{i,n}} - 1 = \frac{S_{i,n+1} - S_{i,n}}{S_{i,n}}$, this gain can also be described as the net return per dollar invested in asset i at time n and realized at time $n+1$.
- This gain/net return can also be invested in the MMA and can be carried forward unchanged to any later date. i.e. one can get $\frac{S_{i,n+1} - S_{i,n}}{S_{i,n}}$ units of the MMA at any later date $n+2, n+3$ etc.

Backward Return

- On the last slide, we gave a financial interpretation to the net return per dollar invested in asset i, viz $\frac{S_{i,n+1}-S_{i,n}}{S_{i,n}}$.
- The division by $S_{i,n}$ is commonly done to remove the scale, allowing one to compare eg. $\frac{S_{2,n+1}-S_{2,n}}{S_{2,n}}$ with $\frac{S_{1,n+1}-S_{1,n}}{S_{1,n}}$ when $S_{2,n}$ and $S_{1,n}$ differ in size.
- So long as price changes are reasonably small, one can alternatively remove the scale by using the backward return, defined as:

$$\frac{S_{i,n+1}-S_{i,n}}{S_{i,n+1}}.$$

- Can we also gave a financial interpretation to this backward return?

Backward Return (Con'd)

- Suppose as before that an investor buys one share of asset i at time n, financing the cost by borrowing $S_{i,n}$ dollars at zero interest rate.
- At time $n+1$, the investor again sells the share and repays the loan, again realizing the gain of $S_{i,n+1} - S_{i,n}$ dollars.
- Now suppose that instead of investing this gain in the MMA, the investor instead uses this gain to buy shares of the i-th asset. The number of shares that can be purchased at time $n+1$ is the backward return $\frac{S_{i,n+1}-S_{i,n}}{S_{i,n+1}} \in \mathbb{R}$.
- This backward return is also the gain on a financed position in the i-th asset, expressed in units of the i-th asset.
- Assuming no dividends on the i-th asset, this backward return can again be carried forward through time unchanged, resulting in a position of $\frac{S_{i,n+1}-S_{i,n}}{S_{i,n+1}}$ shares of the $i-th$ asset at any later date $n+2, n+3$ etc.

Backward Return vs Forward Return

- So far, we have considered 3 trading strategies:
  1. buy 1 share, borrow, and realize into MMA
  2. buy \$1 worth of shares, borrow, and realize into MMA
  3. buy 1 share, borrow, and realize into the i-th risky asset
- The 3rd strategy is perfectly feasible; its gains, measured in units of the $i-th$ asset, are as easy to express as the gains in \$ of the 2nd strategy.
- However, suppose we are forced to express the gains in the 3rd strategy using only the usual forward returns $r_{in} \equiv \frac{S_{i,n+1}-S_{in}}{S_{in}}$. Then, we have that:

$$\frac{S_{i,n+1}-S_{i,n}}{S_{i,n+1}} = \frac{S_{i,n+1}-S_{i,n}}{S_{i,n}(1+r_{in})} = \frac{S_{i,n+1}-S_{i,n}}{S_{i,n}}(1-r_{in}+O(r_{in}^2)),$$

  using the fact that $\frac{1}{1+x} = 1-x+x^2-x^3+\ldots = 1-x+O(x^2)$.
- Expressing the RHS in returns, $\frac{S_{i,n+1}-S_{i,n}}{S_{i,n+1}} - r_{in} = -r_{in}^2 + O(r_{in}^3)$.
- Thus, the leading term in the difference between the P&L of the 3rd strategy, measured in units of the $i-th$ asset, and the forward return, r_{in}, is the negative of the lognormal variance rate r_{in}^2.

Carrying Gains in the Numeraire Portfolio

- Let $L_n > 0$ denote the value of the numeraire portfolio at time n.
- Let's consider a 4th trading strategy, namely buy $\frac{L_n}{S_{in}}$ shares of the i-th asset at time n, borrow the cost, and now realize into the numeraire portfolio.
- Measured in units of the numeraire portfolio, the P&L on such a strategy is $\frac{L_n}{S_{in}} \frac{S_{i,n+1} - S_{i,n}}{L_{n+1}}$.
- When we are forced to express this P&L in terms of the forward returns $r_{in} \equiv \frac{S_{i,n+1} - S_{in}}{S_{in}}$ and $r_{Ln} \equiv \frac{L_{n+1} - L_n}{L_{in}}$, it becomes:

$$\frac{L_n}{S_{in}} \frac{S_{i,n+1} - S_{i,n}}{L_{n+1}} = \frac{r_{in}}{1 + r_{Ln}} = r_{in} - r_{in} r_{Ln} + O(r_{in} r_{Ln}^2).$$

- Hence, with everything expressed in units of the numeraire portfolio, the cost of creating the payoff $r_{in} - r_{in} r_{Ln}$ is $O(r_{in} r_{Ln}^2))$.
- Since this result holds for each asset in the numeraire portfolio, it holds for the numeraire portfolio itself. With everything expressed in units of the numeraire portfolio, the cost of creating the payoff $r_{Ln} - r_{Ln}^2$ is $O(r_{Ln}^3)$.

From Discrete to Continuous

- Recall that with everything expressed in units of the numeraire portfolio, the cost of creating the payoff $r_{in} - r_{in}r_{Ln}$ is $O(r_{in}r_{Ln}^2))$ for each asset.
- This construction was done with zero interest rates for simplicity. When the riskfree rate r_f need not vanish, the same result holds so long as each r_i is replaced by the excess return $r_i - r_f$.
- If we now assume that investors can trade continuously and that all asset prices are positive continuous semi-martingales, then it follows that at each time t, one can realize ex post the gain of $\frac{dS_{it}}{S_{it}} - r_t dt - \frac{d\langle S_i, L\rangle_t}{S_{it}L_t}$ units of the numeraire portfolio without investing any money at t.
- Here, $\frac{dS_{it}}{S_{it}}$ means the total derivative of the Itô integral $\int_0^t \frac{1}{S_{it}} dS_{it}$, while $\frac{d\langle S_i, L\rangle_t}{S_{it}L_t}$ means the total derivative of the integral $\int_0^t \frac{1}{S_{it}L_t} d\langle S_i, L\rangle_t$.
- Letting $\sigma_{iL,t}dt \equiv \frac{d\langle S_i, L\rangle_t}{S_{it}L_t}$, it costs 0 at t to realize a gain ex post of $\frac{dS_{it}}{S_{it}} - (r_t + \sigma_{iL,t})dt$ units of the numeraire portfolio.

Risk Premia Re-interpreted

- Recall it costs 0 to get $\frac{dS_{it}}{S_{it}} - (r_t + \sigma_{iL,t})dt$ units of the numeraire portfolio.
- Suppose that we write each asset's return under $\mathbb{P}$ as:

$$\frac{dS_{it}}{S_{it}} = (r_t + \pi_{it})dt + \sigma_{it} dB_{it}, \qquad t \geq 0,$$

  where $\pi_{it} \in \mathbb{R}$ is usually called the i-th asset's risk premium, σ_{it} is called the i-th asset's volatility, and B_i is a standard Brownian motion under $\mathbb{P}$.
- Suppose that we choose L so that $\sigma_{iL,t} = \pi_{it}$ for $i = 0, 1, \ldots, n$ and hence:

$$\frac{dS_{it}}{S_{it}} - (r_t + \sigma_{iL,t})dt = \sigma_{it} dB_{it}, \qquad t \geq 0.$$

- One can say that each asset's risk premium π_i arises because we carry gains forward using the numeraire portfolio rather than the MMA and because we insist on using Itô integrals rather than backward integrals.

Risk and Reward

- Recall that under no arbitrage, a portfolio with value $L > 0$ was constructed so that the risk premium on each asset arises purely from covariation $\sigma_{iL,t}$ of returns on that asset with the return of the numeraire portfolio:

$$\frac{dS_{it}}{S_{it}} - r_t dt = \sigma_{iL,t} dt + \sigma_{it} dB_{it}, \qquad t \geq 0,$$

  where B_i is a standard Brownian motion under $\mathbb{P}$.

- It follows that the SDE for L under $\mathbb{P}$ is:

$$\frac{dL_t}{L_t} - r_t dt = \sigma_{Lt}^2 dt + \sigma_{Lt} dB_{Lt} \qquad t \geq 0,$$

  where σ_{Lt} is the volatility of the numeraire portfolio and B_L is a standard Brownian motion under $\mathbb{P}$. The risk premium for the numeraire portfolio IS its risk, as measured by σ_{Lt}^2. What could be easier?

- Using Itô's formula, one easily finds that relative prices S_i/L solve:

$$\frac{d(S_{it}/L_t)}{S_{it}/L_t} = \sigma_{it} dB_{it} - \sigma_{Lt} dB_{Lt}, \qquad t \geq 0.$$

- So the probability measure $\mathbb{P}$ is also a martingale measure when the numeraire is Long's numeraire portfolio.

Part IV: Preference-Free Ross Recovery for Bounded Diffusions

- Using Long's numeraire portfolio, we replace Ross's restrictions on the form of preferences with our restrictions on how prices evolve under $\mathbb{Q}$.
- More precisely, we suppose that the prices of some given set of assets are all driven by a univariate time-homogenous bounded diffusion process, X.
- Letting L denote the value of the numeraire portfolio for these assets, we furthermore assume that L is simply a function of X and t and that (X, L) is a bivariate time homogenous diffusion.
- We show that these assumptions determine the $\mathbb{P}$ dynamics of X and all of the spot prices of the assets in the given set.

Our Assumptions

- We assume no arbitrage for some finite set of assets which includes a money market account (MMA).
- As a result, there exists a risk-neutral measure $\mathbb{Q}$ under which spot prices deflated by the MMA balance evolve as martingales.
- We assume that under $\mathbb{Q}$, the driver X is a time homogeneous *bounded* diffusion process:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \qquad t \in [0, T],$$

  where $X_t \in (\ell, u), t \geq 0$, $a(x) > 0$, and where W is SBM under $\mathbb{Q}$.
- We also assume that under $\mathbb{Q}$, the value L of the numeraire portfolio solves:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma_L(X_t)dW_t, \qquad t \in [0, T].$$

- We know the functions $b(x)$, $a(x)$, and $r(x)$ but not $\sigma_L(x)$. How to find it?

Value Function of the Numeraire Portfolio

- Recalling that X is our driver, we assume:

$$L_t \equiv L(X_t, t), \qquad t \in [0, T],$$

where $L(x, t)$ is a positive function of $x \in \mathbb{R}$ and time $t \in [0, T]$.

- Applying Itô's formula, the volatility of L is:

$$\sigma_L(x) \equiv \frac{1}{L(x,t)} \frac{\partial}{\partial x} L(x,t) a(x) = a(x) \frac{\partial}{\partial x} \ln L(x,t).$$

- Dividing by $a(x) > 0$ and integrating w.r.t. x:

$\ln L(x,t) = \int^x \frac{\sigma_L(y)}{a(y)} dy + f(t)$, where $f(t)$ is the constant of integration.

- Exponentiating implies that the value of the numeraire portfolio separates multiplicatively into a positive function $\pi(x)$ of the level x of the driver X and a positive function $p(t)$ of time t:

$$L(x,t) = \pi(x) p(t),$$

where $\pi(x) = e^{\int^x \frac{\sigma_L(y)}{a(y)} dy}$ and $p(t) = e^{f(t)}$.

Separation of Variables

- The numeraire portfolio value function $L(x,t)$ must solve the following linear parabolic PDE to be self-financing:

$$\frac{\partial}{\partial t}L(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}L(x,t) + b(x)\frac{\partial}{\partial x}L(x,t) = r(x)L(x,t).$$

- On the other hand, the last slide shows that this value separates as:

$$L(x,t) = \pi(x)p(t).$$

- Using Bernoulli's classical separation of variables argument, we know that:

$$p(t) = p(0)e^{\lambda t},$$

and that:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

Regular Sturm Liouville Problem

- Recall the ODE on the last slide:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u),$$

  where $\pi(x)$ and λ are unknown.

- Whichever boundary conditions we are allowed to impose, they will be separated. As a result, we have a *regular* Sturm Liouville problem.
- From Sturm Liouville theory, we know that there exists an eigenvalue $\lambda_0 > -\infty$, smaller than all of the other eigenvalues, and an associated positive eigenfunction, $\pi_0(x)$, which is unique up to positive scaling.
- All of the eigenfunctions associated to the other eigenvalues switch signs at least once.
- One can numerically solve for both the smallest eigenvalue λ_0 and its associated positive eigenfunction, $\pi_0(x)$. The positive eigenfunction $\pi_0(x)$ is unique up to positive scaling.

Value Function of the Numeraire Portfolio

- Recall that λ_0 is the known lowest eigenvalue and $\pi_0(x)$ is the associated eigenfunction, positive and known up to a positive scale factor.
- Knowing $\pi_0(x)$ up to positive scaling and knowing λ_0 implies that we also know the value function of the numeraire portfolio up to positive scaling, since:

$$L(x,t) = \pi_0(x)e^{\lambda_0 t}, \qquad x \in [\ell, u], t \in [0, T].$$

- As a result, the volatility of the numeraire portfolio is *uniquely* determined:

$$\sigma_L(x) = a(x)\frac{\partial}{\partial x}\ln \pi_0(x), \qquad x \in [\ell, u].$$

- Mission accomplished! Let's see what the market believes.

$\mathbb{P}$ Dynamics of the Numeraire Portfolio

- Recall that in our diffusion setting, Long (1990) showed that the $\mathbb{P}$ dynamics of L are given by:

$$\frac{dL_t}{L_t} = [r(X_t) + \sigma_L^2(X_t)]dt + \sigma_L(X_t)dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion (SBM) under the probability measure $\mathbb{P}$.

- In equilibrium, the risk premium of the numeraire portfolio is simply $\sigma_L^2(x)$.
- Since we have determined $\sigma_L(x)$ on the last slide, the risk premium of the numeraire portfolio has also been uniquely determined.
- Rewriting the top equation as $\frac{(dL_t/L_t)-r(X_t)dt}{\sigma_L(X_t)} = dB_t + \sigma_L(X_t)dt, t \geq 0$, we see that the process describing the market price of Brownian risk dB_t is simply the known function $\sigma_L(x)$ evaluated at X_t.
- We must have that under $\mathbb{Q}, dW_t = dB_t + \sigma_L(X_t)dt$ is the increment of a $\mathbb{Q}$ SBM. What does this imply about the $\mathbb{P}$ dynamics of X?

$\mathbb{P}$ Dynamics of the Driver

- Recall that the $\mathbb{Q}$ dynamics of X were assumed to be:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \qquad t \geq 0,$$

where recall W is a standard Brownian motion under $\mathbb{Q}$.

- By Girsanov's theorem, W will gain drift $\sigma_L(X_t)$ when we switch measures to $\mathbb{P}$, so that $dB_t = dW_t - \sigma_L(X_t)dt$ is the increment of a $\mathbb{P}$ SBM. Hence, the dynamics of the driver X under the probability measure $\mathbb{P}$ are:

$$dX_t = [b(X_t) + \sigma_L(X_t)a(X_t)]dt + a(X_t)dB_t, \qquad t \geq 0.$$

- Hence, we now know the $\mathbb{P}$ dynamics of the driver X.
- We still have to determine the $\mathbb{P}$ transition density of the driver X.

$\mathbb{P}$ Transition PDF of the Driver

- From the change of numeraire theorem, the Radon Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{-\int_0^T r(X_t)dt} \frac{L(X_T, T)}{L(X_0, 0)} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T} e^{-\int_0^T r(X_t)dt},$$

  since $L(x,t) = \pi_0(x)e^{\lambda_0 t}$.

- Solving for the PDF $d\mathbb{P}$ gives:

$$d\mathbb{P} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T} e^{-\int_0^T r(X_t)dt} d\mathbb{Q} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T} d\mathbb{A}.$$

  where $d\mathbb{A} \equiv e^{-\int_0^T r(X_t)dt} d\mathbb{Q}$ denotes the Arrow Debreu state pricing density.

- Knowing $d\mathbb{Q}$ implies that we also know the Arrow Debreu state pricing density $d\mathbb{A}$, at least numerically. As we also know the positive function $\frac{\pi_0(y)}{\pi_0(x)}$ and the positive function $e^{\lambda_0 T}$, we know $d\mathbb{P}$, the transition PDF under $\mathbb{P}$ of X.

$\mathbb{P}$ Dynamics of Spot Prices

- Also from Girsanov's theorem, the dynamics of the $i-$th spot price S_{it} under $\mathbb{P}$ are uniquely determined as:

$$dS_{it} = [r(X_t)S_i(X_t,t)+\sigma_L(X_t)\frac{\partial}{\partial x}S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}S_i(X_t,t)a(X_t)dB_t,$$

  where for $x \in (\ell, u), t \in [0, T]$, $S_i(x,t)$ solves the following linear PDE:

$$\frac{\partial}{\partial t}S_i(x,t) + \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}S_i(x,t) + b(x)\frac{\partial}{\partial x}S_i(x,t) = r(x)S_i(x,t),$$

  subject to appropriate boundary and terminal conditions. If $S_{it} > 0$, then the SDE at the top can be expressed as:

$$\frac{dS_{it}}{S_{it}} = [r(X_t)+\sigma_L(X_t)\frac{\partial}{\partial x}\ln S_i(X_t,t)a^2(X_t)]dt+\frac{\partial}{\partial x}\ln S_i(X_t,t)a(X_t)dB_t, t \geq 0.$$

- In equilibrium, the instantaneous risk premium is just $d\langle \ln S_i, \ln L\rangle_t$, i.e. the increment of the quadratic covariation of returns on S_i with returns on L.

Part V: Examples of Diffusions on Unbounded Domain

- Our results thus far apply only to diffusions on bounded domains.
- Hence, our results thus far can't be used to determine whether one can uniquely determine $\mathbb{P}$ in models like Black Scholes (1973) and Cox Ingersoll & Ross (1985) (aka CIR), where the diffusing state variable X lives on an unbounded domain such as $(0, \infty)$.
- We don't yet know the general theory here, but we do know two interesting examples of it.

Example 1: Black Scholes Model for a Stock Price

- Suppose that the state variable X is a stock price whose initial value is observed to be the positive constant S_0.
- Suppose we assume or observe zero interest rates and dividends and we assume that the spot price is geometric Brownian motion under $\mathbb{Q}$:

$$\frac{dX_t}{X_t} = \sigma dW_t, \qquad t \geq 0.$$

- Suppose only one stock option trades and from its observed market price, we learn the numerical value of σ.
- All of our previous assumptions are in place, except now we have allowed the diffusing state variable X to live on the unbounded domain $(0, \infty)$.

Example 1: Black Scholes Model (con'd)

- Recall the general ODE governing the positive function $\pi(x)$ & the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- In the BS model with zero rates, $a^2(x) = \sigma^2 x^2$, $b(x) = r(x) = 0$, $\ell = 0$, and $u = \infty$, so we want a positive function $\pi(x)$ and a scalar λ solving the Euler ODE:

$$\frac{\sigma^2 x^2}{2}\pi''(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

- In the class of twice differentiable functions, there are an uncountably infinite number of eigenpairs (λ, π) with π positive. This result implies Ross can't recover here because there are too many candidates for the value of the numeraire portfolio.

- However, all of the positive functions $\pi(x)$ are not square integrable. If we insist on this condition as well, then there are no candidates for the value of the numeraire portfolio. Ross can't recover again, but for a different reason.

Example 2: CIR Model for the Short Rate

- Suppose that after calibrating to caps, floors and swaptions, we find that the short interest rate r solves the following mean-reverting square root process under $\mathbb{Q}$:

$$dr_t = (\mu - \kappa r_t)dt + \sigma\sqrt{r_t}dW_t, \qquad t \geq 0,$$

where r_0, μ, κ, and σ are all known positive constants.

- Recall the general ODE governing the positive function $\pi(x)$ & the scalar λ:

$$\frac{a^2(x)}{2}\pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \qquad x \in (\ell, u).$$

- Here, $a^2(x) = \sigma^2 x > 0$, $b(x) = \mu - \kappa x$, $r(x) = x$, $\ell = 0$, and $u = \infty$ so we want a positive function $\pi(x)$ and a scalar λ solving the linear ODE:

$$\frac{\sigma^2 x}{2}\pi''(x) + (\mu - \kappa x)\pi'(x) - x\pi(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

Example 2: CIR Model (Con'd)

- Recall we want a positive function $\pi(x)$ and a scalar λ solving the ODE:

$$\frac{\sigma^2 x}{2}\pi''(x) + (\mu - \kappa x)\pi'(x) - x\pi(x) = -\lambda\pi(x), \qquad x \in (0, \infty).$$

- The scale density $s(x)$ and speed density $m(x)$ of the CIR process are:

$$s(x) = x^{\frac{-2\mu}{\sigma^2}} e^{\frac{2\kappa}{\sigma^2}x} \qquad m(x) = \frac{2}{\sigma^2} x^{\frac{2\mu}{\sigma^2}-1} e^{\frac{-2\kappa}{\sigma^2}x}.$$

- These densities are used to determine the nature of the boundaries 0 and ∞. As is well known, ∞ is a natural boundary, while 0 is an entrance boundary if $\mu \geq \frac{\sigma^2}{2}$ and regular if $\mu \in (0, \frac{\sigma^2}{2})$. When 0 is an entrance boundary, we must have the reflecting boundary condition $\lim_{x \downarrow 0} \frac{f'(x)}{s(x)} = 0$. When 0 is a regular boundary, we choose to have this condition apply.
- Suppose we consider the weighted square integrable function space $L^2((0,\infty), m(x)dx)$ Then the spectrum is discrete with eigenvalues and eigenfunctions known in closed form (see e.g. Gorovoi and Linetsky (2004)).

Example 2: CIR Model (Con'd)

- Examining the closed form expression for the eigenvalues, we observe that the lowest eigenvalue is $\lambda_0 = \frac{\mu}{\sigma^2}(\gamma - \kappa)$, where $\gamma \equiv \sqrt{\kappa^2 + 2\sigma^2} > \kappa$.
- Examining the closed form expression for the eigenfunctions, we observe that the associated eigenfunction is $\pi_0(x) = e^{-\frac{\gamma-\kappa}{\sigma^2}x}$, which is positive. All of the other eigenfunctions switch signs at least once.
- It follows that in the CIR short rate model, the value function for the numeraire portfolio must be:

$$L(r,t) = \pi_0(r)e^{\lambda_0 t} = e^{-\frac{\gamma-\kappa}{\sigma^2}r + \frac{\mu}{\sigma^2}(\gamma-\kappa)t}, \qquad r > 0, t \geq 0.$$

- Under $\mathbb{P}$, Girsanov's theorem implies that the short rate r solves the SDE:

$$dr_t = (\mu - \gamma r_t)dt + \sigma\sqrt{r_t}dB_t, \qquad t \geq 0,$$

  where B is a standard Brownian motion. Hence, the short rate is still a CIR process under $\mathbb{P}$, but with larger mean reversion since $\gamma > \kappa$.
- In this example, Ross recovery succeeded and moreover we used his model!

Summary

- We highlighted Ross's Theorem 1 and proposed an alternative preference-free way to derive the same financial conclusion.
- Our approach is based on imposing time homogeneity on the $\mathbb{Q}$ dynamics of the value L of Long's numeraire portfolio when it is driven by a bounded time homogeneous diffusion process X.
- We showed how separation of variables allows us to separate beliefs from preferences.
- We explored two examples of unbounded diffusions. In the first (Black Scholes model for stock price), we are unable to recover the $\mathbb{P}$ drift of the stock. In the second (CIR model for the short rate), we were able to recover the $\mathbb{P}$ dynamics of the short rate.
- At present, we do not have a general theory giving sufficient conditions for when Ross recovery succeeds for unbounded diffusions. Considering the widespread use of unbounded diffusions in mathematical finance, this is a good open problem for future research.

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