Weak Reflection Principle and Static Hedging of Barrier Options

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Static vs Dynamic

- In Finance, hedging is a process of offsetting the risks arising from holding a financial instrument by trading (buying and selling) other instruments.
- When the market is complete, the price of a derivative contract can always be replicated by dynamically trading the underlying asset S.
  - Such **dynamic hedging** strategies have certain drawbacks, in particular, due to the presence of transaction costs.
- Whether the market is complete or not, there sometimes exists a **static portfolio** of *simpler (liquid) derivatives*, such that the value of the portfolio **matches the value of the target (exotic) derivative at any time before the barrier is hit**.

Barrier Options: Up-and-Out Put

- Consider the problem of
  - **static hedging** of
  - **barrier options**.

- For the sake of transparency, we focus on the **Up-and-Out Put (UOP)** option:
  - An **UOP** written on the underlying process S is issued with a maturity date $T > 0$, a strike price $K > 0$, and a flat upper barrier $U > K$.
  - At expiry, it **pays off**:

$$I_{\{\sup_{t\in[0,T]} S_t < U\}} \cdot (K - S_T)^+$$

Definition of Static Hedge

- When it exists, a **static hedging** strategy of a barrier option is characterized by a function

$$G : [0, \infty) \to \mathbb{R},$$

  such that a European option with payoff $G(S_T)$ at T **has the same value as the target barrier option** up to and including the time when the barrier is hit.

- One can, then, **hedge the barrier option** by
  1. opening a **long position** in a **European option with payoff** G and
  2. trading it at **zero cost** for the corresponding "vanilla" option when/if the underlying hits the barrier.

Example: Static Hedge in Black's Model

- Consider the **Black's model** where the risk-neutral process for the underlying S is given by a geometric Brownian motion:

$$dS_t = S_t \sigma dW_t,$$

with $S_0 < U$ and $\sigma \in \mathbb{R}$.

- *Carr-Bowie* (1994) show that static hedge of an UOP in such model is given by:

$$G(S) = (K - S)^+ - \frac{K}{U}\left(S - \frac{U^2}{K}\right)^+$$

- Hence an UOP can be replicated exactly by being **long one put** struck at K and **short** $\frac{K}{U}$ **calls** struck at $\frac{U^2}{K}$.

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Static Hedge of an UOP in Black's Model

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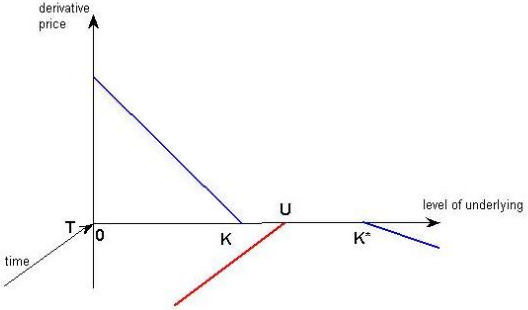

Figure : Static Hedge payoff (blue) and the boundary (red). $K^* = \frac{U^2}{K}$

Related results

- **Explicit exact model-dependent static hedge**
  - BS model: *Bowie-Carr* (1994).
  - Symmetric diffusion models: *Carr-Ellis-Gupta* (1998).
  - Time-homogeneous diffusion models: *Carr-N.* (2011), *Gesell* (2011).

- **Robust sub/superreplicating** static strategies: *Brown-Hobson-Rogers* (2001), *Cox-Hobson-Obłój* (2008), *Cox-Obłój* (2010), *Galichon-Henry-Labordère-Touzi-Obłój-Spoida* (to appear).

- **Robust static hedging with beliefs**: *N.-Obłój* (in progress). We use the exact model-dependent static hedges as building blocks to construct sub- and super replicating strategies that work in classes of models.

- **Optimization-based approach** to find approximate static hedge: *Sachs, Maruhn, Giese, Sircar, Avelaneda* .

Exact static hedge in diffusion models

In *Carr-N.* (2011), we provide **exact static hedging** strategies for barrier options in the following class of models.

- **Pricing of contingent claims is linear**: it is done by taking expectations of discounted payoffs under some pricing measure.
- **Interest rate r is constant**.
- Under the pricing measure, the underlying S follows a **time-homogeneous diffusion**:

$$dS_t = \mu(S_t)dt + \sigma(S_t)dB_t$$

- We make some regularity assumptions on μ and σ. In particular, our results hold for all models where $\sigma(S)/S$ is bounded away from zero, and $\mu(S)/S$ and $\sigma(S)/S$ have limits at the boundary points, and are bounded along with their first three derivatives.

Static Hedge of an UOP

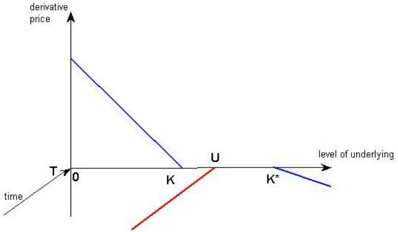

Figure : Static hedge payoff G (blue) and the barrier (red).

Function G **has to be of the form** $G(S) = (K - S)^+ - g(S)$**, where**:

- $g(S) = 0$ for $S < U$,
- and the price of a **European option with payoff** g is equal to the price of a **put with strike** K and the same maturity, whenever the underlying hits the barrier.

"Mirror" Image

Find g, s.t. it has support in (U, ∞) and

$$\mathbb{E}\left[h\left(S_\tau\right) \middle| S_0 = U\right] = \mathbb{E}\left[g\left(S_\tau\right) \middle| S_0 = U\right], \quad \text{for all } \tau > 0$$

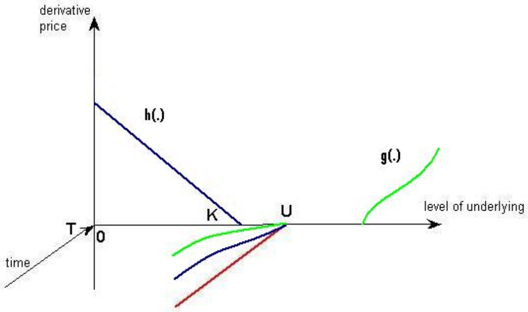

Figure : Price functions of the options with payoffs h (blue) and g (green), along the barrier $S = U$ (red)

Problem formulation

- Consider a stochastic process $X = (X_t)_{t\geq 0}$, started from zero: $X_0 = 0$.
- Introduce
  - Ω_1 – the set of regular functions (e.g. continuous, with at most exponential growth) with support in $(-\infty, 0)$;
  - Ω_2 – the set of regular functions with support in $(0, \infty)$.
- **Problem:** find a mapping $\mathbf{R} : \Omega_1 \to \Omega_2$, such that, for any $f \in \Omega_1$:

$$\mathbb{E}\left[f(X_t)\right] = \mathbb{E}\left[\mathbf{R}f(X_t)\right],$$

  for all $t \geq 0$.

Strong Reflection Principle

- If there exists a mapping $S : \mathbb{R} \to \mathbb{R}$, which maps $(-\infty, 0)$ into $(0, \infty)$, and such that the process X is **symmetric** with respect to this mapping:

$$\text{Law}(S(X_t),\ t \geq 0) = \text{Law}(X_t,\ t \geq 0),$$

then, the reflection $\mathbf{R}$ is easy to construct:

$$\mathbf{R}f = f \circ S$$

- For example, **Brownian motion** B is symmetric with respect to zero:

$$\text{Law}(-B_t)) = \text{Law}(B_t), \qquad \forall t \geq 0,$$

and, therefore:

$$\mathbf{R}f(x) = f(-x)$$

- What we call a **Classical (Strong) Reflection Principle** arises as a combination of the **continuity** of B, its **strong Markov property**, and the above **symmetry**.

Applications

- The **Strong Reflection Principle** for Brownian motion is used
  - to compute the **joint distribution of Brownian motion** and its running maximum:

$$\mathbb{P}(B_T \leq K, \max_{t\in[0,T]} B_t \leq U)$$

  - or, more generally, solve the **Static Hedging problem** when the underlying is a Brownian motion (or any process symmetric with respect to the barrier).
- It turns out that the **Weak Reflection Principle** is enough to solve the above problems.
- We show how to extend this principle to a large class of Markov processes, **which do not possess any strong symmetries!**.

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Weak Reflection Principle for time-homogeneous diffusions

- To simplify the resulting expression, we assume that $\mu \equiv 0$.

$$dX_t = \sigma(X_t) dB_t$$

- Then, the reflection mapping **R** is given by

$$\mathbf{R}h(x) = \frac{2}{\pi i} \int_{\varepsilon - \infty i}^{\varepsilon + \infty i} \frac{z \psi_1(x, z)}{\partial_x \psi_1(0, z) - \partial_x \psi_2(0, z)} \int_{-\infty}^{0} \frac{\psi_1(s, z)}{\sigma^2(s)} h(s) ds\, dz,$$

- where the functions $\psi^1(x, z)$ and $\psi^2(x, z)$ are the **fundamental solutions** of the associated **Sturm-Liouville** equation:

$$\frac{1}{2} \sigma^2(x) \psi_{xx}(x, z) - z^2 \psi(x, z) = 0$$

Solution to the Static Hedging problem

(N.-Carr, 2011)

- Recall that, in order to solve the static hedging problem, we only need to compute the **mirror image of the put payoff**.

$$h(x) = (K - x)^+$$

- Thus, the Static Hedge of an UOP option (with barrier U and strike $K < U$) is given by

$$G(x) = (K - x)^+ - g(x),$$

where

$$g\left(x\right) = \frac{1}{\pi i}\int_{\varepsilon-\infty i}^{\varepsilon+\infty i} \frac{\psi^1\left(x,z\right)\psi^1(K,z)}{\psi^1_x(U,z) - \psi^2_x(U,z)}\frac{dz}{z},$$

Constant Elasticity of Variance: $\mu = 0$, $\sigma(S) = S^{1+\beta}$

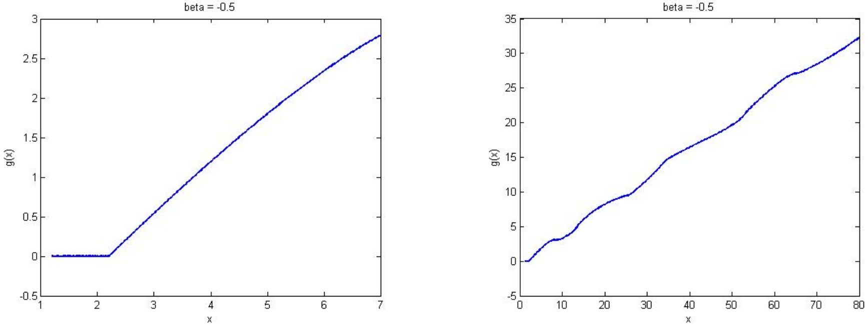

Figure : The "mirror image" g in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the case of $\beta = -0.5$, for small (left) and large (right) values of the argument.

Other CEV: Bachelier and Black-Scholes

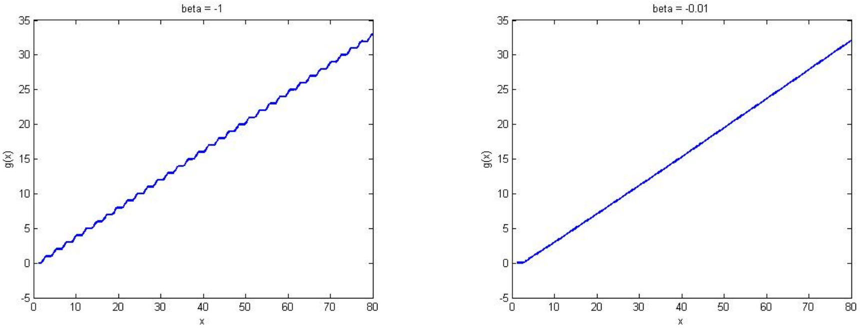

Figure : The "mirror image" g in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the cases of $\beta = -1$ (left) and $\beta \approx 0$ (right).

Computation and extensions

$$g(S) = \frac{1}{\pi i} \int_{\varepsilon - \infty i}^{\varepsilon + \infty i} \frac{\psi^1(S,z)\,\psi^1(K,z)}{\psi^1_S(U,z) - \psi^2_S(U,z)} \frac{dz}{z},$$

- If $\sigma(S)$ is **piece-wise constant**, the fundamental solutions $\psi_1(S,z)$ and $\psi_2(S,z)$ can be easily computed as **linear combinations of exponentials**, on each sub-interval in S.
- This family of models is **sufficient for all practical purposes**.
- The proposed static hedge also succeeds in all models that arise by running the time-homogeneous diffusion on an **independent continuous stochastic clock**.
- One can obtain a **semi-robust extension of this static hedging strategy**. More precisely, a strategy that succeeds in all models, as long as the market implied volatility stays within given bounds (beliefs about implied volatility are fulfilled).

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Robust static hedge with beliefs on implied volatility

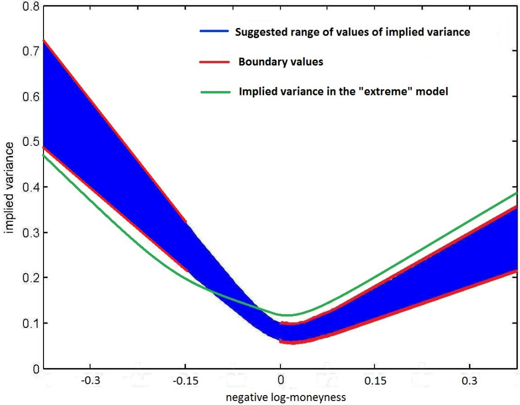

Figure : Range of possible values of (beliefs on) implied volatility (blue), and the extremal implied volatility produced by a diffusion model (green)

Weak Reflection Principle for Lévy processes with one-sided jumps

- Note that, in principle, this method can be used any **strong Markov process** which **does not jump across the barrier**.
- The only problem is to establish the **Weak Reflection** of this process with respect to the barrier.
- In our ongoing work, we have developed the **Weak Reflection principle for Lévy processes with one-sided jumps**. The solution takes a similar for, in the sense that the reflection operator **R** is given as an integral transform, with a kernel that can be computed through the characteristic exponent of the Lévy process ψ. For example, the image of $h(y) = \mathbf{1}_{\{y \leq K\}}$, for $K < 0$, is given by

$$\mathbf{R}h(y) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{e^{\varepsilon y + iwy + iKz}}{z} \left(\frac{\psi'(w)}{\psi(w) - \psi(-iz)} - \frac{1}{w + iz} \right) dz dw$$

Beyond Finance

- The Weak Reflection Principle allows to compute the **joint distribution** of a time-homogeneous diffusion and its **running maximum**, through the marginal distribution of the process itself:

$$\mathbb{P}(X_T \le K, \max_{t\in[0,T]} X_t \le U) = \mathbb{E}(g^{K,U}(X_T)),$$

  where $g^{K,U}$ is the mirror image of function $(K-.)^+$, with respect to the barrier U.

- The **connection to PDE's** yields various applications in **Physics** and **Biology**.

Summary

- I have presented a solution to the **Static Hedging** problem for barrier options.
- This solution provides **exact**, but **model-dependent**, hedge in all regular enough time-homogeneous diffusion models.
- In our ongoing work with J. Obloj, we develop the **semi-robust** hedges based on the above results.
- Static Hedging problem motivated the development of a new technique, the **Weak Reflection Principle**.
- We have developed the Weak Reflection principle to diffusion processes and one-sided Lévy processes.

Summary (cont'd)

- The **Weak Reflection Principle** allows us to
  - control the expected value of a function of the process,
  - at any time when the process is at the barrier of a given domain,
  - by changing the function outside of this domain.
- Applications include **Finance, Physics, Biology, Computational Methods**.
- Further extensions:
  - **Specific applications in Physics and Biology?**
  - **More general domains**?
  - **More general stochastic processes**?

Non-existence result

- *Bardos-Douady-Fursikov* (2004) treat this problem for a general parabolic PDE, and prove the existence of **approximate solutions** g_ε, such that

$$\sup_{t\in[0,T]} \left| u^h(U,t) - u^{g_\varepsilon}(U,t) \right| < \varepsilon$$

- They show that an exact solution doesn't exist in general...
- Their proof is **not constructive** - finding even an approximate solution is left as a separate problem.
- The example of **non-existence** relies heavily on the **time-dependence of the coefficients** in the corresponding PDE!

Naive numerical approximation

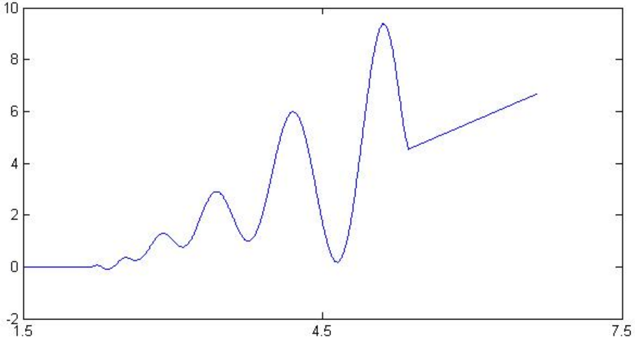

Figure : Payoff function g_ε as a result of the naive least-square optimization approach

Square root process revisited

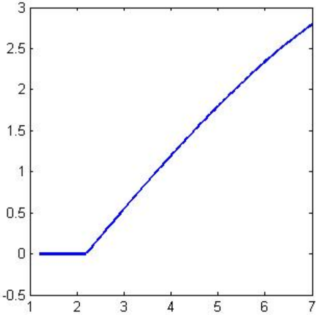

Figure : 2. Function g, for $\beta = -0.5$, $U = 1.2$, $K = 0.5$

Notice that there is a constant $K^* \geq U$, such that the support of g is exactly $[K^*, \infty]$.

Short-Maturity Behavior and Single-Strike Hedge

- **Key observation: when time-to-maturity is small, only the values of g around K^* matter**!
- Thus, for small maturities, the approximation of the payoff function g with a **scaled call payoff** should perform well.
- We have:

$$\int_K^U \frac{dy}{\sigma(y)} = \int_U^{K^*} \frac{dy}{\sigma(y)}, \qquad \eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}}.$$

- Using the above, we can construct the **single-strike sub- and superreplicating strategies**: there exists $\delta > 0$, such that, whenever $S_t = U$,

$$[1 - \delta(T-t)]\, P_t(K) - \eta C_t(K^*) \leq 0 \leq [1 + \delta(T-t)]\, P_t(K) - \eta C_t(K^*)$$

Function g: properties and numerical computation

- There exists a constant $K^* \geq U$, such that the support of g is exactly $[K^*, \infty]$.
- Introduce the "*signed geodesic distance*":

$$Z(x) := \sqrt{2} \int_U^x \frac{dy}{\sigma(y)}$$

- Then K^* is a solution of the equation

$$Z(K^*) + Z(K) = 0$$

- The function g is "*analytic with respect to the geodesic distance Z*" in (K^*, ∞):

$$g(x) = \sum_{k=1}^{\infty} c_k \left(Z(x) - Z(K^*) \right)^k ,$$

and the exists an algorithm for computing c_k's.