MIP reformulations of some chance-constrained mathematical programs

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joint work with Ahmad Abdi

Outline

1. Introduction
2. Our results
   - Characterizing valid inequalities
   - Separation over a large class of inequalities
   - Explicit facet-defining inequalities
   - Extended formulation
3. Conclusion

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$$\text{(LP)} \quad \min \quad c^T x \qquad \text{s.t.} \quad Ax \leq b$$

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  - Chance-constrained program: Want a high probability of satisfying set of constraints

Chance-constrained programs

Chanced-constrained linear programming problem with stochastic right-hand sides:

$$
\begin{array}{lll}
\text{(PLP)} & \min & c^T x \\
 & \text{s.t.} & \mathbb{P}(Tx \geq \xi) \geq 1 - \epsilon \\
 & & x \in X,
\end{array}
$$

- $X \subset \mathbb{R}^d$: a polyhedron,
- ξ: random variable in $\mathbb{R}^m$ with finite discrete distribution,
- $T \in \mathbb{R}^{m \times d}$,
- $\epsilon \in (0, 1)$, and
- $c \in \mathbb{R}^n$.

Suppose that ξ takes values from $\xi^1, \ldots, \xi^n$ with probabilities $\pi_1, \ldots, \pi_n$, respectively.

Feasibility: Example

Consider the special case where $d = m = 3$, $n = 4$, $c = (1, 1, 1)$, $X = \mathbb{R}^d$, $T = I$, $\epsilon = 0.5$, $\pi = (0.25, 0.25, 0.25, 0.25)$.

Let $\{\xi^j\}_{j=1}^n = \left\{ \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$

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Then $x = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$ is not feasible, since it $Tx = x \not\geq \xi^j$, for $j = 1, 2, 3$.

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Then x is feasible if and only if $x \geq \xi^j$ for at least two $j \in \{1, \ldots, 4\}$.
So to get a feasible solution, one needs to pick any two out of 4 scenarios to be satisfied.
Thus

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \mathbb{P}(Tx \geq \xi) \geq 1 - \epsilon \\ & x \in X, \end{array} \Rightarrow \quad \begin{array}{ll} \min & \sum_{i=1}^{m} \max_{j \in I} \xi_i^j \\ \text{s.t.} & I \subseteq \{1, \ldots, n\}, |I| \geq 2 \end{array}$$

NP-hardness [Luedtke, Ahmed and Nemhauser, 2010]

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \max_{j \in I} \xi_i^j \\ \text{s.t.} \quad & I \subseteq \{1, \ldots, n\}, |I| \geq K \end{aligned}$$

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$\xi_v^e = 1$ if edge e is incident to vertex v, 0 otherwise.

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So if there is a clique of size $C(C-1)/2$, then there is a solution to such problem with $K = C(C-1)/2$ and objective $\leq C$ (and vice-versa). □

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- Nonetheless, the discrete distribution case with Monte-Carlo sampling and using as confidence parameter $\alpha > \epsilon$ gives us a lower bound on the optimal solution with probability $1 - exp\{-2n(\alpha - \epsilon)^2\}$ (Luedtke et al, 2010).

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- Nonetheless, the discrete distribution case with Monte-Carlo sampling and using as confidence parameter $\alpha > \epsilon$ gives us a lower bound on the optimal solution with probability $1 - exp\{-2n(\alpha - \epsilon)^2\}$ (Luedtke et al, 2010).
- Under some mild assumptions (like continuity), the discrete distribution case with Monte-Carlo sampling and using as confidence parameter $\alpha < \epsilon$ gives us a feasible solution with probability approaching 1 exponentially with n (Luedtke et al, 2010).

Previous work on Chance-constrained programs

- Charnes et al. (1958) - Chance-constrained programs with disjoint probabilistic constraints
- Miller and Wagner (1965) - Chance-constrained programs with joint probabilistic constraints for independent random variables.
- Prékopa (1970) - Log-concave probability measure on random right-hand sides: There is an equivalent convex program
- Sen (1992) - Chance-constrained programs with discrete distributions using disjunctive programming reformulation.
- Ruszczynski (2002) - Precedence-constrained knapsack set: Use scenarios that dominate others.
- Luedtke, Ahmed and Nemhauser (2010) and Küçükyavuz (2012) - Chance-constrained programs with discrete random variables in the right-hand-side: MIP reformulation
- Our work: Extends Luedtke et al. (2010) and Küçükyavuz (2012)

MIP reformulation [Luedtke, Ahmed and Nemhauser, 2010]

Let $z \in \{0,1\}^n$ where $z_j = 0$ guarantees that $Tx \geq \xi_j$. Then (PLP) is equivalent to

$$
\begin{array}{lll}
\text{(PLP)} & \min & c^T x \\
& \text{s.t.} & y = Tx \\
& & y + z_j \xi_j \geq \xi_j \quad \forall j \in [n] \\
& & \sum_{j=1}^n \pi_j z_j \leq \epsilon \\
& & z \in \{0,1\}^n \\
& & x \in X.
\end{array}
$$

Let's look at a single chance-constraint: For each $k \in [m]$, let

$$\mathcal{D}_k := \left\{ (y_k, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{j=1}^n \pi_j z_j \leq \epsilon,\ y_k + \xi_{jk} z_j \geq \xi_{jk}\ \forall j \in [n] \right\}.$$

Goal

Study $\mathcal{D}_k$ and get cuts to strengthen the MIP formulation

Changing notation a bit:

$$\mathcal{Q} := \left\{ (y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{j \in [n]} a_j z_j \leq p,\ y + h_j z_j \geq h_j\ \forall j \in [n] \right\}.$$

Other related work

Mixing set with a knapsack constraint:

$$\mathcal{Q} := \left\{ (y,z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{j\in[n]} a_j z_j \le p,\ \ y + h_j z_j \ge h_j \ \forall j \in [n] \right\}.$$

- Work on the mixing set:
  - Günlük and Pochet
  - Zhao and de Farias,
  - Conforti, di Summa and Wolsey
  - Miller and Wolsey
  - van Vyve
  - to mention a few
- Qiu, Ahmed, Dey, Wolsey - Cover LPs with violation
- J. Luedtke, S. Küçükyavuz, Y. Song, A Branch-and-Cut Algorithm for the Chance-Constrained Knapsack Problem.
- S. Küçükyavuz, M. Zhang, On Continuous Mixing Set with a Cardinality Constraint.

Cutting plane approach

Mixed Integer Programming (MIP):

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{aligned}$$

1. Let P_I be the set of feasible solutions to our MIP.
2. Let R be a relaxation of P_I (typically by dropping integrality constraints), that is $R \supseteq P_I$.
3. Solve the optimization problem over R, with optimal solution $x^* \in R$
4. Find a cut/cutting plane/valid inequality: An inequality $\pi^T x \leq \pi_o$ satisfied by all $x \in P_I$, but with $\pi^T x^* > \pi_o$
5. Let $R \leftarrow R \cap \{x : \pi^T x \leq \pi_o\}$
6. Go to 3

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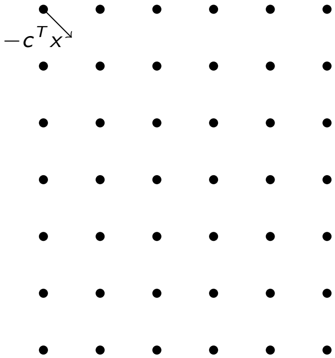

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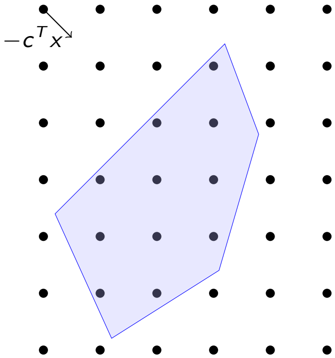

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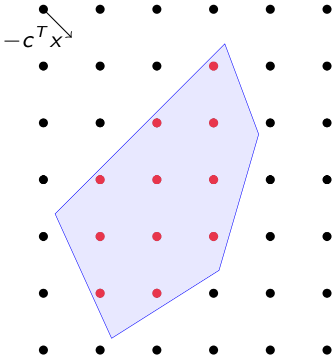

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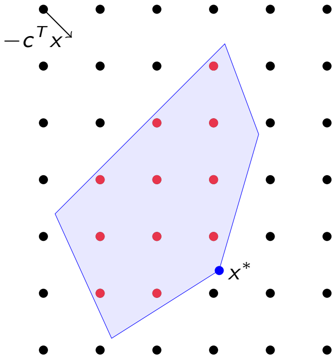

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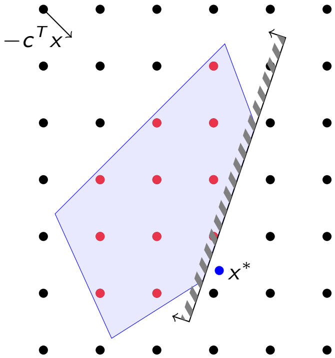

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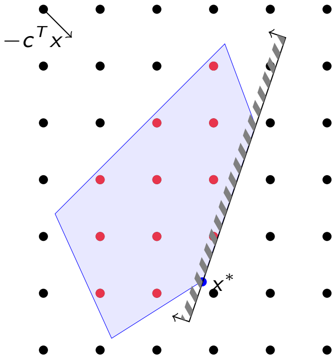

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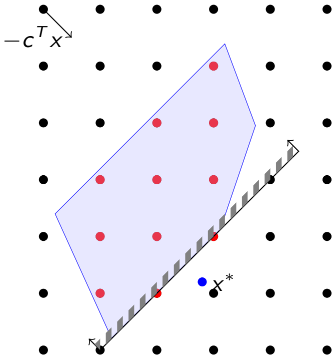

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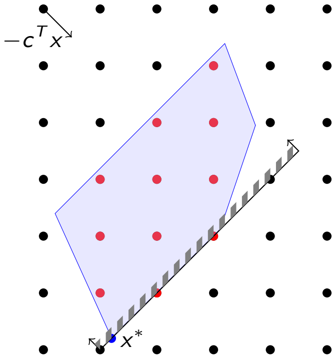

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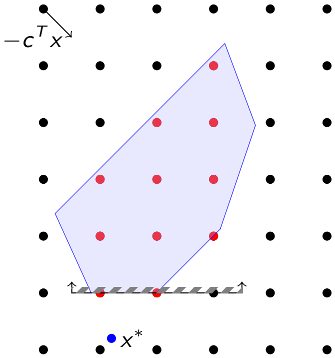

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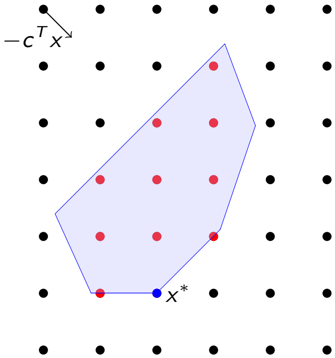

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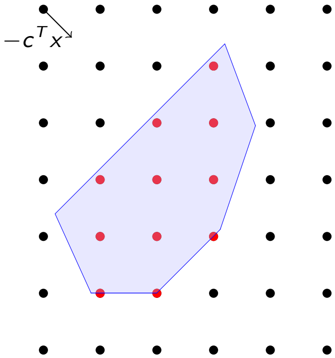
Valid inequalities/
Cutting planes/
Cuts



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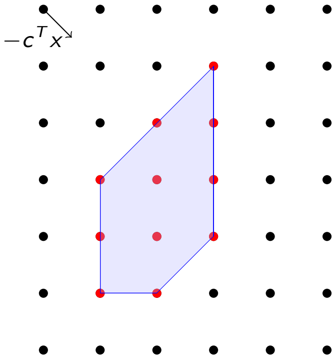

Valid inequalities/
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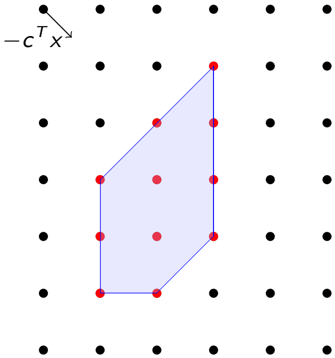
Want "strongest possible" valid inequalities (facet-defining): Get the convex hull of feasible solutions

Cutting plane approach

Mixed Integer Programming (MIP):

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$



$$\pi^1 x \leq \pi_o^1$$
$$\pi^2 x \leq \pi_o^2$$
$$\pi^3 x \leq \pi_o^3$$

Valid inequalities/
Cutting planes/
Cuts

Want "strongest possible" valid inequalities (facet-defining): Get the convex hull of feasible solutions

OBS: A strong formulation may be obtained implicitly, for instance as a projection of a higher dimensional polyhedron (Extended formulation).

Mixing set with a knapsack constraint

Luedtke et al. (2010) and Küçükyavuz (2012) study inequalities for

$$\mathcal{Q} := \left\{ (y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{j \in [n]} a_j z_j \leq p, \;\; y + h_j z_j \geq h_j \;\forall j \in [n] \right\}.$$

- For the cardinality constraint/equal probabilities case (all $a_j = 1$):
  - Characterization of all valid inequalities and separation (Luedtke et al.)
  - Extended formulation (Luedtke et al.)
  - (T, Π_L) inequalities (Küçükyavuz)
  - Compact extended formulation (Küçükyavuz)
- For the general constraint/probabilities case:
  - Strengthened star inequalities (Luedtke et al.)
  - Extended formulation for LP+strengthen star (Luedtke et al.)
  - Strengthened (T, Π_L) inequalities (Küçükyavuz)

Mixing set with a knapsack constraint

Experiments by Küçükyavuz:

Table 2 Probabilistic lot-sizing experiments

f	τ	Gap	% Gapimp			Cuts			Nodes			Time (endgap)		
			CPX	Mix	TL	CPX	Mix	TL	CPX	Mix	TL	CPX	Mix	TL
0	5	1.3	23	90	90	85	559	199	1,139	117	81	130	255	59
	10	1.7	26	97	97	131	833	333	10,493	171	68	821	529	104
	15	2.0	24	98	98	217	1360	559	36,765	265	209	T(0.4)	1,387	341
	20	2.4	18	97	97	248	1,527	779	25479.0	291	418	T(1.0)	1,764	967
1	5	2.4	47	74	74	235	574	359	72,889	3,258	15,663	T(0.3)	T(0.4)	3,476 (0.2)
	10	2.7	43	78	77	288	846	499	39,773	1,323	4,169	T(0.7)	T(0.5)	T(0.4)
	15	3.1	39	77	76	373	1,334	707	22,837	492	1,831	T(1.4)	T(0.7)	T(0.6)
	20	3.5	36	75	76	452	1,849	1,089	17,031	245	939	T(1.7)	T(0.9)	T(0.7)

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Our goal: Extend their results

Outline

Valid inequalities for Q

Lemma

Suppose that

$$\gamma y + \sum_{j \in [n]} \alpha_j z_j \geq \beta \qquad (1)$$

is a valid inequality for $\text{conv}(\mathcal{Q})$ *for some* $\alpha \in \mathbb{R}^n, \gamma, \beta \in \mathbb{R}$*. Then* $\gamma \geq 0$*. Moreover, if* $\gamma = 0$ *then* (1) *is a valid inequality for* $\mathcal{P} := \{z \in \{0,1\}^n : \sum_{j=1}^{n} a_j z_j \leq p\}$.

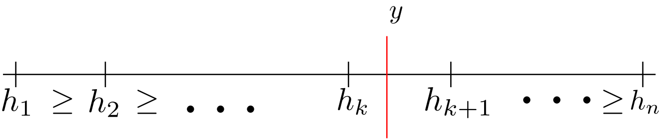
Proof.

Observe that $(1, 0) \in \text{rcone}(\text{conv}(\mathcal{Q}))$. This implies that $\gamma \geq 0$. Moreover, since $\text{conv}(\mathcal{P}) = \text{proj}_z(\text{conv}(\mathcal{Q}))$, it follows that if $\gamma = 0$ then (1) is a valid inequality for $\text{conv}(\mathcal{P})$. □

A few observations

$$\mathcal{Q} := \left\{ (y,z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{j\in[n]} a_j z_j \le p,\ \ y + h_j z_j \ge h_j \ \forall j \in [n] \right\}.$$

- We may assume, WLOG that $h_1 \ge h_2 \ge \ldots h_n$.
- Let $\nu := \max\{k : \sum_{j\le k} a_j \le p\}$.
- If $h_k > y \ge h_{k+1}$, then $z_j = 1$ for all $j \le k$
- Therefore, $y \ge h_{\nu+1}$



Inequalities that do not come from $\mathcal{P}$

Previous Lemma implies we may assume have the form:

$$y + \sum_{j \in [n]} \alpha_j z_j \geq \beta$$

Define, for each $0 \leq k \leq \nu$, the knapsack set

$$\mathcal{P}_k := \left\{ z \in \{0,1\}^{[n]} : \sum_{j > k} a_j z_j \leq p - \sum_{j \leq k} a_j \right\}.$$

Theorem (Abdi and F.)

$y + \sum_{j \in [n]} \alpha_j z_j \geq \beta$ is a valid inequality for conv($\mathcal{Q}$) if and only if $(\alpha, \beta) \in \mathcal{G}$, where

$$\mathcal{G} := \left\{ (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j \leq k} \alpha_j + \sum_{j > k} \alpha_j z_j^* + h_{k+1} \geq \beta, \ \forall \ z^* \in \mathcal{P}_k, \forall \ 0 \leq k \leq \nu. \right\}$$

Rewriting $\mathcal{G}$

$$\mathcal{G} := \left\{ (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j \le k} \alpha_j + \sum_{j > k} \alpha_j z_j^* + h_{k+1} \ge \beta, \ \forall\ z^* \in \mathcal{P}_k, \forall\ 0 \le k \le \nu. \right\}$$

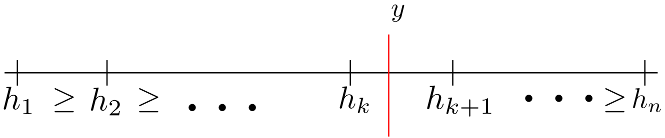
Let

$$f_k(\alpha) := \min \left\{ \sum_{j > k} \alpha_j z_j : z \in \mathcal{P}_k \right\}. \qquad (2)$$

Then it is easy to see that

$$\mathcal{G} = \left\{ (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j \le k} \alpha_j + f_k(\alpha) + h_{k+1} \ge \beta, \ \forall\ 0 \le k \le \nu \right\}.$$

Proof

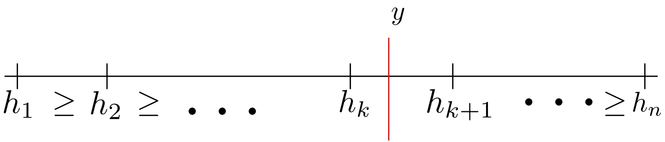


Recall:
$z_j = 1$ for all $j \leq k$

($\Leftarrow$) Let $(y^*, z^*) \in \mathcal{Q}$. Let $h_0 := +\infty$.
Then $h_k > y^* \geq h_{k+1}$ for some $0 \leq k \leq \nu$. Since $z_j^* = 1$ for all $j \leq k$, $z^* \in \mathcal{P}_k$. Then

$$
\begin{aligned}
y^* + \sum_{j \in [n]} \alpha_j z_j^* &\geq h_{k+1} + \sum_{j \leq k} \alpha_j + \sum_{j > k} \alpha_j z_j^* \\
&\geq h_{k+1} + \sum_{j \leq k} \alpha_j + f_k(\alpha) \\
&\geq \beta \quad \text{since } (\alpha, \beta) \in \mathcal{G}
\end{aligned}
$$

Proof



Recall:
$z_j = 1$ for all $j \leq k$

($\Rightarrow$) Take $0 \leq k \leq \nu$. Let $z^* \in \{0,1\}^n$ be an optimal solution to $f_k(\alpha)$ with $z_j^* = 1$ for all $1 \leq j \leq k$. Let $y^* = h_{k+1}$. Observe that $(y^*, z^*) \in \mathcal{Q}$. Hence, validity implies that

$$\begin{aligned}\beta &\leq y^* + \sum_{j \in [n]} \alpha_j z_j^* \\ &= h_{k+1} + \sum_{j \leq k} \alpha_j + \sum_{j > k} \alpha_j z_j^* \\ &= h_{k+1} + \sum_{j \leq k} \alpha_j + f_k(\alpha).\end{aligned}$$

Since this holds for all $0 \leq k \leq \nu$, it follows that $(\alpha, \beta) \in \mathcal{G}$.

A few observations

- This theorem implies that separation over inequalities of this form can be done in polytime if optimization over $\mathcal{P}$ can be solved in polytime

$$f_k(\alpha) := \min \left\{ \sum_{j>k} \alpha_j z_j : z \in \mathcal{P}_k \right\}.$$

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- Note that if $\alpha \geq 0$, $f_k(\alpha)$ can be solved easily for all k

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- Note that if $\alpha \geq 0$, $f_k(\alpha)$ can be solved easily for all k
- Complexity of separation "resides" in negative coefficients.

$$f_k(\alpha) := \min \left\{ \sum_{j>k} \alpha_j z_j : z \in \mathcal{P}_k \right\}.$$

Separation over a large class of inequalities

Let R, S be a partition of $[n]$. Let's assume that $j \in R$ implies $\alpha_j \geq 0$. Then any inequality of the form

$$y + \sum_{j \in [n]} \alpha_j z_j \geq \beta$$

with $\alpha_j \geq 0$, for all $j \in R$ is in $\mathcal{G} \cap \{\alpha_j \geq 0, \forall j \in R\}$

Theorem (Abdi and F.)

If optimization over $\{z \in \{0,1\}^S : \sum_{j \in S} a_j z_j \leq p\}$ *can be done in polynomial time, then separation over all valid inequalities* $y + \sum_{j \in [n]} \alpha_j z_j \geq \beta$ *such that* $\alpha_j \geq 0, \forall j \in R = [n] \setminus S$ *can be done in polynomial-time.*

Examples:

- a_j not too large for $j \in S$
- All a_j equal for $j \in S$ (in this case, a better characterization is possible)

Theorem (Abdi and F.)

Consider an inequality of the form

$$y + \sum_{j \in [n]} \alpha_j z_j \geq \beta. \tag{3}$$

If there exists $(\sigma, \rho) \in \mathbb{R}_-^{\nu+1} \times \mathbb{R}_-^{(\nu+1)(n-\nu/2)}$ *such that* $(\alpha, \beta, \sigma, \rho) \in \mathcal{J}$*, then* (3) *is a valid inequality for* conv($\mathcal{Q}$)*. Furthermore, when* $a_i = 1$ *for all* $i \in S \subseteq [n]$ *and* (3) *is a valid inequality for* conv($\mathcal{Q}$) *with* $\alpha_j \geq 0, \forall j \in R := [n] \setminus S$*, then there exists* $(\sigma, \rho) \in \mathbb{R}_-^{\nu+1} \times \mathbb{R}_-^{(\nu+1)(n-\nu/2)}$ *such that* $(\alpha, \beta, \sigma, \rho) \in \mathcal{J}$*, where*

$$\mathcal{J} := \left\{ (\alpha, \beta, \sigma, \rho) : \begin{array}{ll} \sum_{j \leq k} \alpha_j + \beta_k \sigma_k + \sum_{j > k} \rho_{kj} + h_{k+1} \geq \beta & \forall\ 0 \leq k \leq \nu, \\ a_j \sigma_k + \rho_{kj} \leq \alpha_j & \forall\ n \geq j > k \\ & \forall\ 0 \leq k \leq \nu. \\ \sigma, \rho \leq 0, \alpha_j \geq 0, \forall j \in R & \end{array} \right\}$$

Proof idea

We saw that the set of valid inequalities is characterized by

$$f_k(\alpha) := \min \left\{ \sum_{j>k} \alpha_j z_j : z \in \mathcal{P}_k \right\}$$

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Use such characterization and LP duality to obtain the formulation.

Properties of facet-defining inequalities for $conv(\mathcal{Q})$

Lemma (Abdi and F.)

Suppose that the inequality

$$y + \sum_{j \in [n]} \alpha_j z_j \geq \beta \tag{4}$$

is facet-defining for $conv(\mathcal{Q})$*. Then*

(i) (α, β) *is an extreme point of* $\mathcal{G}$*,*

(ii) $\beta = h_1 + f_0(\alpha)$*, and*

(iii) *if* $\alpha_k < 0$ *for some* $1 \leq k \leq n$*, then* $a_k > 0$*.*

Note: $f_0(\alpha) := \min \left\{ \sum_{j \in [n]} \alpha_j z_j : \sum_{j \in [n]} a_j z_j \leq p \right\}$

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- Consider the cardinality constrained case ($a_j = 1, \forall j$).

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- Suppose $|S| \leq p$
- Then $f_0(\alpha) := \min \left\{ \sum_{j \in [n]} \alpha_j z_j : \sum_{j \in [n]} a_j z_j \leq p \right\} = \sum_{j \in S} \alpha_j$, hence any such facet-defining inequality will have the form:

$$y + \sum_{j \in R} \alpha_j z_j + \sum_{j \in S} \alpha_j z_j \geq h_1 + \sum_{j \in S} \alpha_j$$

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$$y + \sum_{j\in R} \alpha_j z_j + \sum_{j\in S} \alpha_j z_j \geq h_1 + \sum_{j\in S} \alpha_j$$

$$y + \sum_{j\in R} \alpha_j z_j + \sum_{j\in S} \Delta_j (1 - z_j) \geq h_1$$

with $\alpha, \Delta \geq 0$

Luedtke et al.'s inequalities for the cardinality constrained case:

Theorem (Luedtke et al.)

Let $m \in \{1, \ldots, p\}$*,* $R = \{r_1, \ldots, r_a\} \subseteq \{1, \ldots, m\}$ *and* $S = \{q_1, \ldots, q_{p-m}\} \subseteq \{p+1, \ldots, n\}$*. For* $m < p$*, define* $\Delta_1^m = h_{m+1} - h_{m+2}$ *and*

$$\Delta_i^m = \max\left\{\Delta_{i-1}^m, h_{m+1} - h_{m+i+1} - \sum_{j=1}^{i-1} \Delta_j^m\right\}, i = 2, \ldots, p-m$$

Then, with $h_{r_{a+1}} = h_{m+1}$*,*

$$y + \sum_{j=1}^{a} (h_{r_j} - h_{r_{j+1}}) z_{r_j} + \sum_{j=1}^{p-m} \Delta_j^m (1 - z_{q_j}) \geq h_{r_1}$$

is valid for $\mathcal{Q}$ *and facet-defining if* $h_{r_1} = h_1$

Validity proof

Note: $0 \leq \Delta_1^m \leq \Delta_2^m \leq \ldots \leq \Delta_{p-m}^m$

$$\mathcal{G} := \left\{ (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j \leq k} \alpha_j + f_k(\alpha) + h_{k+1} \geq \beta, \forall\ 0 \leq k \leq \nu. \right\}$$

In this case, $\nu = p$. Note that $S \subseteq \{p+1, \ldots, n\}$, so $j \in S \Rightarrow j > \nu = p$.

$$f_k(\alpha) := \min \left\{ \sum_{j > k} \alpha_j z_j : z \in \mathcal{P}_k \right\}$$

$$f_k(\alpha) := \min \left\{ \sum_{j \in S} -\Delta_j^m z_j : \sum_{j \in S} z_j \leq p - k \right\} = - \sum_{j=k-m+1}^{p-m} \Delta_j^m$$

$$\sum_{j\le k}\alpha_j + f_k(\alpha) + h_{k+1} \ge \beta$$

becomes

$$\sum_{j\le k: j\in R}\alpha_j - \sum_{j=k-m+1}^{p-m}\Delta_j^m + h_{k+1} \ge h_1 - \sum_{j\in S}\Delta_j^m \iff$$

$$\sum_{j\le k: j\in R}\alpha_j + \sum_{j=1}^{k-m}\Delta_j^m + h_{k+1} \ge h_1$$

Note that $\sum_{j\le k:j\in R}\alpha_j = h_1 - h_{r_t}$ for some $r_t > k$.

$$\sum_{j\le k: j\in R}\alpha_j + \sum_{j=1}^{k-m}\Delta_j^m + h_{k+1} \ge h_1$$

So if $k < m$

$$\sum_{j\le k: j\in R}\alpha_j + \sum_{j=1}^{k-m}\Delta_j^m + h_{k+1} = h_1 - h_{r_t} + h_{k+1} \ge h_1$$

And if $k \ge m$, $\Delta_{k-m}^m \ge h_{m+1} - h_{m+k-m+1} - \sum_{j=1}^{k-m-1}\Delta_j^m$ so

$$\sum_{j\le k: j\in R}\alpha_j + \sum_{j=1}^{k-m}\Delta_j^m + h_{k+1} \ge \sum_{j\in R}\alpha_j + h_{m+1} - h_{k+1} + h_{k+1} = h_1 - h_{r_{a+1}} + h_{m+1} = h_1$$

More inequalities for the cardinality constrained case

Theorem (Küçükyavuz, 2012)

Suppose that $a_j = 1$ *for all* $j \in [n]$*. Take a positive integer* $m \le \nu = p$*. Suppose that*

(i) $R := \{r_1, \ldots, r_a\} \subset \{1, \ldots, m\}$, *where* $r_1 < \ldots < r_a$*; and*

(ii) $S \subset \{m+2, \ldots, n\}$ *and take a permutation of the elements in* S*,*
$\Pi_S := \{q_1, \ldots, q_{p-m}\}$ *such that* $q_j > m + j$ *for all* $1 \le j \le p - m$*.*

Set $r_{a+1} := m + 1$*. Let* $\Delta_1 := h_{m+1} - h_{m+2}$*, and for* $2 \le j \le p - m$ *define*

$$\Delta_j := \max\left\{\Delta_{j-1}, h_{m+1} - h_{m+1+j} - \sum(\Delta_i : q_i > m + j, i < j)\right\}.$$

Then the (T, Π_L) *inequality*

$$y + \sum_{j=1}^{a}(h_{r_j} - h_{r_{j+1}})z_{r_j} + \sum_{j=1}^{p-m}\Delta_j(1 - z_{q_j}) \ge h_{r_1} \tag{5}$$

is valid for $\mathrm{conv}(\mathcal{Q})$*. Furthermore, (5) is facet-defining inequality for* $\mathrm{conv}(\mathcal{Q})$ *if and only if* $h_{r_1} = h_1$*.*

Theorem (Abdi and F.)

Take an integer $0 \leq m \leq \nu$ *such that* $p - s_m$ *is an integer. For each* $1 \leq j \leq p - s_m$*, let* $k(j) := \max\{k : j \geq s_k - s_m\}$*. Let*

(i) $R := \{r_1, \ldots, r_a\} \subset \{1, \ldots, m\}$ *where* $r_1 < \ldots < r_a$*;*

(ii) $S := \{q_1, \ldots, q_s\} \subset \{m+2, \ldots, n\}$ *where* $s = p - s_m$ *and* $q_j > k(j)$ *for all* $1 \leq j \leq p - s_m$*; and*

(iii) S *is chosen so that* $a_j = 1$ *for all* $j \in S$*, and* $a_j \leq s_m$ *for all* $j \notin S$*.*

Set $r_{a+1} = m + 1$*. Let* $\Delta_{q_1} := h_{m+1} - h_{k(1)+1}$*, and for* $2 \leq j \leq p - s_m$*, define*

$$\Delta_{q_j} := \max\left\{\Delta_{q_{j-1}}, h_{m+1} - h_{k(j)+1} + \sum\left(\Delta_{q_i} : q_i > k(j), i < j\right)\right\}. \tag{6}$$

Then

$$y + \sum_{j=1}^{a}(h_{r_j} - h_{r_{j+1}})z_{r_j} + \sum_{i \in S}\Delta_i(1 - z_i) \geq h_{r_1} \tag{7}$$

is a valid inequality for $conv(\mathcal{Q})$*. Furthermore, (7) is a facet-defining inequality for* $conv(\mathcal{Q})$ *if and only if* $h_{r_1} = h_1$*.*

Extended formulation

Kücükyavuz (2012):

- If (y^*, z^*) is an extreme point of conv$(\mathcal{Q})$, then $y \in \{h_1, h_2, \dots, h_{\nu+1}\}$.
- For each $0 \leq k \leq \nu$, let

$$\mathcal{Q}_{k+1} := \left\{ (y, z) \in \{h_{k+1}\} \times \{0,1\}^n : \sum_{j \in [n]} a_j z_j \leq p, z_j \geq 1 \ \forall\, j \leq k \right\}.$$

$$\mathcal{R}_{k+1} := \left\{ (y, z) \in \{h_{k+1}\} \times [0,1]^n : \sum_{j \in [n]} a_j z_j \leq p, z_j \geq 1 \ \forall\, j \leq k \right\}.$$

- $conv(\mathcal{Q}) = conv(\bigcup_{k=1}^{\nu+1} \mathcal{Q}_k) + \mathcal{C} \subseteq conv(\bigcup_{k=1}^{\nu+1} \mathcal{R}_k) + \mathcal{C}$
- This leads to a polynomial-size extended formulation that is exact in the cardinality constraint case.

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$$\mathcal{Q}_{k+1} := \left\{ (y, z) \in \{h_{k+1}\} \times \{0, 1\}^n : \sum_{j \in [n]} a_j z_j \leq p, z_j \geq 1 \ \forall \ j \leq k \right\}.$$

$$\mathcal{R}_{k+1} := \left\{ (y, z) \in \{h_{k+1}\} \times [0, 1]^n : \sum_{j \in [n]} a_j z_j \leq p, z_j \geq 1 \ \forall \ j \leq k \right\}.$$

- $conv(\mathcal{Q}) = conv(\bigcup_{k=1}^{\nu+1} \mathcal{Q}_k) + \mathcal{C} \subseteq conv(\bigcup_{k=1}^{\nu+1} \mathcal{R}_k) + \mathcal{C}$
- This leads to a polynomial-size extended formulation that is exact in the cardinality constraint case.
- How strong is such extended formulation in the general case?

Extended formulation

Theorem (Abdi and F.)

Let EQ be the extended formulation obtained as described. Then

$$proj_{y,z}(EQ) \subseteq \left\{ (y,z) \in \mathbb{R}_+ \times [0,1]^n : \begin{array}{ll} y + h_j z_j \geq h_j & ,\forall j = 1,\ldots,n \\ \sum_{j=1}^{n} a_j z_j \leq p & \\ y + \sum_{j=1}^{n} \alpha_j z_j \geq \beta & ,\forall (\alpha,\beta) \in \mathcal{G}_1 \end{array} \right\}$$

where $\mathcal{G}_1$ is the set of coefficients of valid inequalities that are only allowed to have negative coefficients for all j such that $a_j = 1$.

Note: Using similar arguments, one can derive polynomial size extended formulations which have even better provable strength.

A better extended formulation

Bienstock (2008) gave polynomial-size formulations that have arbitrarily close gap to the convex hull of 0-1 solutions to the knapsack set.
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Theorem (Abdi and F.)

For any $M > 0$ and $\epsilon > 0$ there exists a polyhedron $\mathcal{RQ}$ such that conv$(\mathcal{Q}) \subset \mathcal{RQ}$, and if $(y^*, z^*) \in \mathcal{RQ}$ and $y + \sum_{j=1}^n \alpha_j z_j \geq \beta$ is a valid inequality of conv$(\mathcal{Q})$ whose coefficients, α_j, are bounded below by $-M$, then

$$y^* + \sum_{j=1}^{n} \alpha_j z_j^* \geq \beta - \epsilon M n.$$

Moreover, $\mathcal{RQ}$ can be described explicitly as the projection of a polyhedron of dimension $O\left(\epsilon^{-1} n^{1+\lceil 1/\epsilon \rceil} \nu\right)$ that is described by $O\left(\epsilon^{-1} n^{2+\lceil 1/\epsilon \rceil} \nu\right)$ constraints.

Computational experiments

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Outline

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Take away: Try to reduce the complexity of the negative part!

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 and try to increase that; realize the negative side
 and try to reduce.
 That's the way."
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THANK YOU!