Analysis Lecture Series #1: Sequences, Cardinality and Continuity

This handout contains "exercises", which are questions directly related to the lectured material, usually technical matters or examples that the reader should attempt to clarify his/her understanding. It also contains "problems", which are usually harder and are often pulled from contests (so, in particular, the reader should not get discouraged because many of the problems are intended to be hard). They also follow the flow of the lecture, so the reader should consult the lecture notes if he/she would like some insight into the order of presentation of the exercises/problems. Some of the problems have solutions given in the notes. The reader should attempt them first before looking at the solution, if possible.

1 Sequences and Cardinality

Exercise 1.1. Fix $N \in \mathbb{N}$ and let $\mathcal{F}(N) := \{\frac{a}{b} : 0 \leq a < b \leq N, \gcd(a,b) = 1\}$ be the set of Farey fractions of order N. Let $\frac{a}{b}, \frac{c}{d} \in \mathcal{F}(N)$ with $\frac{a}{b} < \frac{c}{d}$.
a) Show that $\mathbb{Q} \cap [0,1] = \bigcup_{N\geq 1} \mathcal{F}(N)$.
b) Prove the mediant property: $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Deduce that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive elements in $\mathcal{F}(N)$ then $b + d > N$.
c) Prove that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in $\mathcal{F}(N)$ if, and only if, $bc - ad = 1$. (Hint: an inductive argument might help). Deduce from this and the previous exercise that

$$\left|\frac{a}{b} - \frac{c}{d}\right| \leq \min\left(\frac{1}{b(N+1-b)}, \frac{1}{d(N+1-d)}\right) < \frac{2}{N+1}.$$

Problem 1.1. Though, strictly speaking, unrelated to the task at hand, we provide a challenging, number theoretical interlude for the reader about Farey sequences, namely giving a bound on the number of elements in $\mathcal{F}(N)$. The reader may want to return to this problem after having studied sequences and series.
i) Prove that $1 < \sum_{n\geq 1} \frac{1}{n^2} < 2$.
ii) Let $\phi(n) := \{1 \leq a \leq n : \gcd(a,n) = 1\}$. Show that $|\mathcal{F}(N)| = 1 + \sum_{n\leq N} \phi(n)$ (since $0 = 0/1$ is in there).
iii) Show that $\phi(p^k) = p^k - p^{k-1}$ whenever p is a prime and $k \geq 1$. Also, show that $\phi(p^k q^l) = \phi(p^k)\phi(q^l)$. Deduce that $\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m,n) = 1$, and that $\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.
iv) Let $\mu(n)$ be the Möbius function, defined by $\mu(n) = 0$ if n is divisible by the square of any prime, and otherwise, $\mu(n) = (-1)^k$, where k is the number of distinct prime factors of n. Show (relating divisors of integers without square factors to subsets of the set of primes dividing n and using the principle of inclusion/exclusion) that $\sum_{d|n} \mu(d) = 0$ if $n > 1$, and $\mu(1) = 1$. Using iii), prove that $\phi(n) = \sum_{d|n} \frac{\mu(d)}{d}$, where the sum runs through all of the divisors of n.
v) Observe that $\left|\sum_{n\geq 1} \frac{\mu(n)}{n^2}\right| \leq \sum_{n\geq 1} \frac{1}{n^2}$. Using iv), show that $\left(\sum_{n\geq 1} \frac{\mu(n)}{n^2}\right)\left(\sum_{n\geq 1} \frac{1}{n^2}\right) = 1$. Deduce from this and the rest of the exercise that $\frac{1}{4}N^2 < |\mathcal{F}(N)| < \frac{1}{2}N^2$. (Hint: the formula for the sum $\sum_{k=1}^{N} k$ may be helpful). Note that the non-triviality in this exercise is that $|\mathcal{F}(N)| > \frac{1}{4}N^2$ since at most half of the pairs in the square $\{1, \ldots, N\} \times \{1, \ldots, N\}$ represent a rational whose denominator is larger than its numerator. In fact, $|\mathcal{F}(N)|$ is about $\frac{3}{\pi^2}N^2$, the improved constant coming from an exact value for the sum in i).

Exercise 1.2. Prove that if n is not a square, $\frac{1}{\sqrt{n}} \notin \mathbb{Q}$.

Exercise 1.3. Let $x, y \in \mathbb{R}$. Check that: i) $|x| = 0$ if, and only if, $x = 0$; ii) $|xy| = |x||y|$; iii) $|x + y| \leq |x| + |y|$. Clearly, iii) implies that $|x+y| \leq 2\max|x|, |y|$. Property iii) is called the triangle inequality. A function that satisfies these iii) properties is called a *generalized absolute value*.
Also, prove the following equivalent form of the triangle inequality: $||x| - |y|| \leq |x - y|$.
As a side note, suppose $\|\|$ is a function on $\mathbb{R}$ that satisfies i) and ii) above, and $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for any $x, y \in \mathbb{R}$. Prove by induction that $\|n\| \leq \|1\|$ for any integer n, evaluate, in two ways, $\|(x+1)^n\|$, where $x \in \mathbb{Q}$. (We will see this absolute value come up later on. In contrast to the regular absolute value, which is called *Archimedean*, this type of absolute value is called *non-Archimedean*.)

Exercise 1.4. Recall that a Cauchy sequence is a sequence $\{b_n\}_n$ such that for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for any $m, n \geq N$, $|a_m - a_n| < \epsilon$. Let $\{b_n\}_n$ be a Cauchy sequence that is not equivalent to the zero sequence. Show that $\{\frac{1}{b_n}\}_n$ is also Cauchy. Deduce that if $\{a_n\}_n$ is also Cauchy then so is $\{\frac{a_n}{b_n}\}_n$. (Hint: the argument invokes the addition and multiplication theorems for Cauchy sequences.)

Exercise 1.5. Recall that two sets satisfy the relation $|A| \leq |B|$ if and only if there exists an injective map from A to B. Given a collection $\mathcal{C}$ of sets, check that $\leq$ is a reflexive and transitive relation, i.e., that $|A| \leq |A|$ and if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$. Note that it is evidently not an equivalence relation. On the other hand, check that the relation $|A| = |B|$, relating to the equicardinality of sets A and B, is an equivalence relation.

Exercise 1.6. Recall that a set A is said to be countable if there is a bijection between A and $\mathbb{N}$, and uncountable otherwise.
i) Prove that the set of all finite subsets of a countable set is countable.
ii) Deduce from i) that the set of real numbers that are roots of an integer polynomial is countable. This will be of interest in a later set of lectures.
iii) Prove that if $|A| = |B|$ then if $\mathcal{P}(A)$ denotes the power set of A (the set of all subsets of A) then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$, and hence deduce that given any countable set A, $\mathcal{P}(A)$ is uncountable.
iv) Let $x, y \in (0, 1)$, and write their decimal expansions $x = \sum_{j\geq 1} a_j 10^{-j}$ and $y = \sum_{j\geq 1} b_j 10^{-j}$. Let $f(x, y) := 0.a_1 b_1 a_2 b_2 \ldots$. Prove that f is an injective map. Also, prove that if $(0, 1)$ is uncountable then (a, b) is uncountable. Finally, observe that $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ is a bijection, so deduce from this, or otherwise, that $\mathbb{R}^2$ and hence $\mathbb{R}^n$ are both equicardinal to $\mathbb{R}$.

Exercise 1.7. Check that the limit of a sequence of real numbers is unique, i.e., if L and L' are both limits of $\{a_n\}_n$ then the definition implies that $L = L'$.

Problem 1.2 (Putnam 1955, B6)**.** Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function that satisfies $\lim_{n\to\infty} f(n) = 0$. Prove that the set of solutions (x, y, z) in positive integers to the equation $f(x) + f(y) + f(z) = 1$ is finite.

Problem 1.3. Prove that $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$ is a well-defined, expression and find its value. (Hint: express it as the limit of a sequence, proving that it converges.)

Exercise 1.8. You will need the definition of the limit supremum and limit infimum in this exercise.
a) Check that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$, for any sequence $\{a_n\}_n$. b) Check also that $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$.

Problem 1.4. In several of the following problems, the reader should familiarize himself/herself with the example of sequences provided by series, as given in the notes.
a) (Euler) Prove that the sequence $\{\sum_{m=0}^{n} \frac{1}{m!}\}_n$ converges (where $0! = 1$ by convention). Let $e := \sum_{n\geq 1} \frac{1}{n!}$. Prove that e is irrational. (Hint: if not then $qe = p \in \mathbb{N}$, for some positive integers p and q. Now use the convergence of the defining series for e to derive a contradiction.)
b) Use the argument in a) to prove the following more general theorem of Erdős: let $\{n_k\}_k$ be a strictly increasing sequence of positive integers such that if $N_k := n_{k+1}^{-1} \prod_{j=0}^{k} n_j$ for each k then $N_k \to \infty$. Prove that $\sum_{k\geq 1} \frac{1}{n_k}$ converges and is irrational. (Note that when $n_j := j!$, we get a)).
c) (Putnam 1981 A1) Let $E(n)$ denote the largest integer k such that 5^k is a divisor of $\prod_{j=1}^{n} j^j$. Find $\lim_{n\to\infty} \frac{E(n)}{n^2}$.
d) (Chris Small's Putnam Archive, 2001) Let $k \in \mathbb{N}$. Determine all $c \in \mathbb{R}$, such that for any sequence $\{x_n\}_n \subset \mathbb{R}$ satisfying $\frac{x_{n+1}+x_{n-1}}{2} = cx_n$, we have $x_{n+k} = x_n$. (Hint: what does periodicity imply about the boundedness of the sequence, and what does the recurrence imply about maximal and minimal elements?)
e) Recall that for $\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}$ sets of real numbers, the Cauchy-Schwarz inequality states that

$$\left|\sum_{k=1}^{n} x_k y_k\right|^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Prove the following generalization: given sequences $\{x_n\}_n$ and $\{y_n\}_n$ of real numbers, prove that

$$\left|\sum_{k\geq 1} x_k y_k\right|^2 \leq \left(\sum_{k\geq 1} x_k^2\right)\left(\sum_{k\geq 1} y_k^2\right).$$

This form is sometimes referred to as Bunyakovsky's inequality.

Exercise 1.9. Let P be a polynomial with real coefficients and suppose that $a_n \to L$. Check that $P(a_n) \to P(L)$.

2 Continuity and Topology on $\mathbb{R}$

Exercise 2.1. You will need the definition of an open and closed set in this exercise.
Check that if I_1 and I_2 are open sets then $I_1 \cap I_2$ is also open. Deduce that any finite intersection of open sets is open. On the other hand, find an *infinite* sequence of open neighbourhoods around zero whose intersection is not open (in fact, the intersection is $\{0\}$).
Let A be a closed set and suppose $\{a_n\} \subset A$ with $a_n \to a \in \mathbb{R}$. Prove that $a \in A$. Hence, any sequence in a closed set converges inside that set.

Exercise 2.2. Let $-\infty < a < b < \infty$ and suppose $f : (a,b) \to \mathbb{R}$ is convex, in the sense that if $x, y \in \mathbb{R}$ then $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y))$. Prove that this property is equivalent to the property that for any $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ (Hint: Think binary.). Hence, or otherwise, prove that f is continuous on (a,b). What happens if we let a or b be infinite?

Problem 2.1. a) (Putnam 1947 A2) Find all continuous function $f : \mathbb{R} \to \mathbb{R}$, such that $f\left(\sqrt{x^2+y^2}\right) = f(x)f(y)$ for each $x, y \in \mathbb{R}$. (Hint: In general, contest problems involving continuous functional equations can be proven by considering the equation for a dense subset of $\mathbb{R}$.)
b) (Putnam 1996 A6) Let $c > 0$ be constant. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = f(x^2 + c)$ for each $x \in \mathbb{R}$. (Hint: Set $g(x) := x^2 + c$ and consider the sequence $\{g^n(x)\}_n$, for x fixed.)
c) (IMO 1972) Suppose $f, g : \mathbb{R} \to \mathbb{R}$, such that $f(x+y) + f(x-y) = 2f(x)g(y)$, for each $x, y \in \mathbb{R}$. Suppose that $f \not\equiv 0$ and that for each $x \in \mathbb{R}$, $|f(x)| \leq 1$. Prove that $|g(y)| \leq 1$ for every $y \in \mathbb{R}$. (Strictly speaking, this can be proven without continuity, but it is interesting to think about continuous functions that satisfy it).
d) Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous function that satisfies $f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a,b)$. Prove that there is a continuous function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(a,b) = g(ab)$. (It is enough to consider continuity of $f(x,y)$ as a function of two variables as the condition that for any x_0 and y_0 fixed, $f(x_0, y)$ is continuous in y and $f(x, y_0)$ is continuous in x.)

Problem 2.2. (L. Larson, 6.2.10) Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$, and let $n \in \mathbb{N}$. Prove that there there is always some $x = x(n) \in [0, x - 1/n]$ such that $f(x) = f(x + 1/n)$.

Exercise 2.3. You will need the definitions of compactness and uniform continuity in this exercise.
a) Show that $f(x) := x^2$ is not uniformly continuous on $(0, \infty)$.
b) Check that any singleton $\{x\}$ is compact. Deduce that it is not necessarily the case that $f^{-1}(K)$ is compact when K is compact if f is any continuous function. A continuous map that does have this property is called a *proper map.*
c) Check that if K_1 and K_2 are both compact then $K_1 \cup K_2$ is compact. Deduce that any finite union of compact sets is compact.