A Review of strong types

Here T will be complete theory. We will work in a monster $\mathcal{C}$ and, as usual, small means of cardinality smaller than the one of $\mathcal{C}$. A sequence in $\mathcal{C}$ is not necessarily finite, but its length is always small.

Definable without more precision means definable without parameters. If we want to precise the set of parameters over which it is definable, we will say it is A-definable or definable over A. A type-definable set (or relation) is a set (or a relation) which is a (possibly infinite, but always small) intersection of definable sets. A set is A-invariant (A is a set of parameters) if it is left fixed by any A-automorphism. An invariant set is a set which is $\emptyset$-invariant.

Remark: A set which is type-definable over B and A-invariant is in fact type-definable over A.

Main references

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B. Kim and A. Pillay, Simple theories, Annals of Pure and Applied Logic, 1997.

D. Lascar, The category of models of a complete theory, Journal of Symbolic Logic 47 (1982), 249-266.

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The 3 strong types

Let A be a set of parameters and a and b be sequence in $\mathcal{C}$. Then

Definitions:

1. The invariant strong type (or the Lascar strong type): $tf_I(a/A) = tf_I(b/A)$ if and only if, for all A-invariant A-bounded equivalence relation R, we have $R(a, b)$.
2. The compact strong type (or the Kim-Pillay strong type): $tf_K(a/A) = tf_K(b/A)$ if and only if, for all A-type-definable A-bounded equivalence relation R, we have $R(a, b)$.
3. The definable strong type (or simply strong type, or the Shelah strong type): $tf(a) = tf(b)$ if and only if, for all finite equivalence relation R definable without parameter, we have $R(a, b)$.

4. If n is an finite or infinite ordinal (but small), we will denote by $S^I_\alpha(A)$ the set M^α divided by the relation $tf_I(x/A) = tf_I(y/A)$. We write $S^I_\alpha(T)$ for $S^I_\alpha(\emptyset)$.

5. Same thing for $S^K_\alpha(A)$

Remarks:

1. It is clear that 1 is stronger than 2 which is stronger than 3.

2. To have the same invariant strong type is itself a invariant bounded equivalence relation, finer that every other; an analogous remark holds for the compact strong type.

¿From now on, we will work with the strong types over the empty set. But, of course, everything that we will say will generalise for strong types over a small set of parameters.

Definition: A theory is G-compact if, for all finite A, the relation $tf_I(a/A) = tf_I(b/A)$ is type definable.

Some motivations

1. Amalgamation of for strong type

2 $\langle Aut_A(\mathcal{C}) \cup Aut_B(\mathcal{C})\rangle = Aut_{A\cap B}(\mathcal{C})$

3. Canonical basis are hyperimagianries in simple theories.

Thick relations

Definition A binary relation R is *thick* if it is symmetric, reflexive and any set of elements which are pairwise not linked by R has small cardinality; a formula φ is thick if it defines a thick relation.

Remarks:

1. If φ is a thick formula, then there is a $k \in \omega$ such that in any set of cardinality bigger than k, there are two distinct points a and b such that $\varphi(a, b)$.

2. If a and b are members of the same infinite indiscernible sequence, then, for all invariant thick relation R, $R(a, b)$.

3. If for every thick formula φ, we have $\varphi(a,b)$, then there exists an infinite indiscernible sequence which contains a and b.

If a and b are members of the same infinite indiscernible sequence, we will write $\Theta(a,b)$. So Θ is the strongest invariant thick relation and it is type-definable.

Facts:

1. If, for some model $M \prec \mathcal{C}$, $t(a/M) = t(b/M)$, then there exists c such that $\Theta(a,c)$ and $\Theta(b,c)$ (i.e. $(a,b) \in \Theta^2$).
2. If $\Theta(a,b)$, then, for some model M, $t(a/M) = t(b/M)$.
3. The transitive closure of the relation Θ is exactly the relation "to have the same invariant strong type". It is also the transitive closure of the relation: there exists a model M such that $t(a/M) = t(b/M)$.

Topology

Let $X \subset \mathcal{C}^\alpha$, and assume that X is invariant. Then X is completely determined by the following set:

$$\begin{aligned} T_p(X) = \{p \in S(T) \quad ; \\ &\text{there exists } a \in X \text{ such that } p = t(a)\} \\ &= \{p \in S(T) \quad ; \quad \text{for all } a \text{ such that } p = t(a),\ a \in X\} \end{aligned}$$

and conversely.

Fact: the following are equivalent:

1. X is type-definable;
2. $T_p(X)$ is a closed subset of $S_\alpha(\mathcal{C})$;
3. $T_p(X)$ is closed under ultraproducts.

Definition: Let $Y \subseteq S^I_\alpha(T)$ or $Y \subseteq S^K_\alpha(T)$ and call X the set of realisations of types in Y (in fact, $X = \bigcup Y$). Then we say that Y is closed if X is type-definable over some small set.

Facts:

1. X (in $S^I_\alpha(T)$ or in $S^K_\alpha(T)$) is closed if and only if it is closed under ultraproducts;

2. $S^K_\alpha(T)$ is a compact Hausdorff space;

3. $S^I_\alpha(T)$ is a compact space (which means that if an intersection of a family closed sets is empty, then an intersection of a finite subfamily is already empty). It is Hausdorff if and only if it is T_1 (that is, all points are closed); if T is G-compact, then $S^I_\alpha(T)$ is Hausdorff.

4. The natural surjections from $S^I_\alpha(T)$ onto $S^K_\alpha(T)$ and from $S^K_\alpha(T)$ onto $S_\alpha(T)$ are continuous.

Galois groups

Definition: The group of I-strong automorphisms of $\mathcal{C}$ is defined by

$$\begin{aligned} Autf_I(\mathcal{C}) = \{g \in Aut(\mathcal{C}) \quad ; \\ \text{for all } \alpha,\ g \text{ leaves pointwise fixed the set } S^I_\alpha(T)\} \end{aligned}$$

In other word, an automorphism g belongs to $Autf_I(\mathcal{C})$ if and only if, for all invariant bounded equivalence relation R and for all sequence a of the right length, we have $\models R(a, g(a))$.

Similarly, we define the group of K-strong automorphisms of $\mathcal{C}$

$$\begin{aligned} Autf_K(\mathcal{C}) = \{g \in Aut(\mathcal{C}) \quad ; \\ \text{for all } \alpha,\ g \text{ leaves pointwise fixed the set } S^K_\alpha(T)\} \end{aligned}$$

They are normal subgroups of $Aut(\mathcal{C})$ and the Galois groups are just the quotient groups:

$$Gal_I(T) = Aut(\mathcal{C})/Autf_I(\mathcal{C})$$
$$Gal_K(T) = Aut(\mathcal{C})/Autf_K(\mathcal{C})$$

We will denote by μ the canonical homomorphism from $Aut(\mathcal{C})$ onto $Gal_I(T)$ and μ' the canonical homomorphism from $Aut(\mathcal{C})$ onto $Gal_K(T)$. There is also a natural homomorphism from $Gal_I(T)$ onto $Gal_K(T)$ that we will denote by ν.

Remarks:

1. Of course $Autf_I(\mathcal{C}) \subseteq Autf_K(\mathcal{C})$. If T is G-compact, then $Autf_I(\mathcal{C}) = Autf_K(\mathcal{C})$.

2. $Gal_I(T)$ can be identified naturally with the group of elementary permutation of $S^I_\alpha(T)$. That is why we allowed ourself to drop the $\mathcal{C}$. The same remark applies of course to the other Galois groups.

Facts:

1. If M is a small submodel of $\mathcal{C}$, then any automorphism which leaves M pointwise fixed belongs to $Autf_I(\mathcal{C})$.

2. If g and h are 2 automorphisms such that, for some m enumerating a submodel M, $t(g(m)/M) = t(h(m)/M)$, then $\mu(g) = \mu(h)$.

3. The cardinality of $Gal_I(T)$ is at most $2^{\|T\|}$. The same is of course true for $Gal_K(T)$.

4. Let $\Delta = \bigcup_{M \prec \mathcal{C}} Aut_M(\mathcal{C})$. Then $Autf_I(\mathcal{C})$ is the subgroup of $Aut(\mathcal{C})$ generated by Δ.

5. If for some $k \in \omega$, $Autf_I(\mathcal{C}) = \Delta^k$, then $Autf_I(\mathcal{C}) = Autf_K(\mathcal{C})$ and invariant strong types and compact strong types are the same thing. I do not know about the converse.

6. If T is stable, then $Autf_I(\mathcal{C}) = \Delta^2$. If T is simple, then $Autf_I(\mathcal{C}) = \Delta^3$.

7. For every $k > 0$, there is a theory such that $Autf_I(\mathcal{C}) = \Delta^k$ but $Autf_I(\mathcal{C}) \neq \Delta^{k-1}$.

8. (Kim) $Autf_I(\mathcal{C}) = Autf_K(\mathcal{C})$ if and only if the 2 following conditions are satisfied: 1. $Autf_I(\mathcal{C})$ is closed in $Aut(\mathcal{C})$ (endowed with the pointwise topology); for each $n \in \omega$, the relation $tf_I(x) = tf_I(y)$ for sequences x and y in M^n is type definable.

Topology on Galois groups

Proposition: Let $C \subseteq Gal_I(T)$. then the following conditions are equivalent:

1. For all sequence a in $\mathcal{C}$, the set
$$\{g(a) \quad ; \ \mu(g) \in C\}$$
is type-definable over some small set;

2. For all sequence a in $\mathcal{C}$, for all $M_0 \prec \mathcal{C}$, the set $\{g(a) \quad ; \ \mu(g) \in C\}$ is type-definable over M_0.

3. C is closed under ultraproducts *to be explained*.

We define a topological structure on $Gal_I(T)$ by decreeing that C is closed if and only if it satisfies the above condition.

We have a similar proposition for $Gal_K(T)$, and we may define a topology on $Gal_K(T)$ in a similar way:

Proposition:Let $C \in Gal_K(T)$. then the following conditions are equivalent:

1. For all sequence a in $\mathcal{C}$, the set

$$\{g(a) \quad ; \mu'(g) \in C\}$$

is type-definable over a small set;

2. For all sequence a in $\mathcal{C}$, for all $M_0 \prec \mathcal{C}$, the set $\{g(a) \quad ; \mu(g) \in C\}$ is type-definable over M_0.

3. C is closed under ultraproducts.

Facts:

1. With this topologies, $Gal_I(T)$ and $Gal_K(T)$ are topological groups;

2. The map μ from $Aut(\mathcal{C})$ endowed with the pointwise topology onto $Gal_I(T)$ is continuous. The same is true for μ' and ν.

3. $Gal_K(T)$ is a compact Hausdorff topological group.

4. If $g \in Gal_I(T)$, the topological closure of $\{g\}$ is exactly $\nu^{-1}\nu(g)$.

5. $Gal_I(T)$ is a compact topological group. It is Hausdorff if and only if it is T_1 if and only if it is equal to $Gal_K(T)$.

6. $Gal_K(T)$ is a profinite group if and only if the compact strong types are equal to the (ordinary) strong types if and only if, on every complete type, every type-definable bounded equivalence relation is the intersection of a family of finite definable equivalence relations.

A Galois correspondence

Let define an ultraimaginary element as being the class of some sequence modulo an invariant equivalence relation. Such an element will be bounded if it has only a small number of conjugate under the action of $Aut(\mathcal{C})$.

We can develop a kind of Galois correspondance between the subgroups of $Gal_I(T)$ and the bounded ultraimagianry elements:

Facts:

1. If e is a bounded ultraimaginary element, then $Aut_e(\mathcal{C})$, the set of automorphisms which leave fixed e, is a subgroup of $Aut(\mathcal{C})$ which contains $Autf_I(\mathcal{C})$

   (so $H = Aut_e(\mathcal{C})/Autf_I(\mathcal{C})$ is a subgoup of $Gal_I(T)$).

2. Conversely, if H is a subgroup of $Gal_I(T)$, then there exists an ultraimaginary element e such that $Aut_e(\mathcal{C}) = \mu^{-1}(H)$.

3. In this correspondance, H is a closed subgroup of $Gal_I(T)$ if and only if e is hyperimaginary (up to interdefinability).
4. And H is clopen if and only if e is an imaginary element.

Using this correspondance and (more deeply) the fact that any compact Hausdorff group is the projective limit of compact Lie group, we can show that any hyperimaginary element is interdefinable with a set of hyperimaginaries of the form a/E where $a \in \mathcal{C}^n$ for some $n \in \omega$ and E a type-definable equivalence relation on $\mathcal{C}^n$.

An example of non G-compact theory

(due to Martin Ziegler)

First we construct, for any $n \in \omega$, a theory T_n. We will see that, if $n = 2k$, then $Autf_I(\mathcal{C}) = \Delta^k$ but $Autf_I(\mathcal{C}) \neq \Delta^{k-1}$.

The language contains one symbol for a ternary relation R and one symbol for a unary function f.

The axioms says that:

1. R is a "dense circular ordering", i.e.:

   $\forall x \forall y \forall z (R(x,y,z) \iff R(y,z,x) \iff R(z,x,y))$;

   $\forall x$("the binary relation $R(x,y,z)$ is a dense linear ordering without endpoints on the set of elements different of x")
2. f respects the relation R;

   $\forall x (f^n(x) = x)$;

   $\forall x (R(f^i(x), f^j(x), f^\ell(x))$ for $0 \leq i \leq j \leq \ell < n$.

Given 2 models M and N of T, it is easy to see that the family of all isomorphisms from a finite substructure of M onto a finite substructure of N has the back and forth property. Consequently, T_n is complete, $\aleph_0$-categorical and has the elimination of quantifiers.

Facts:

1. $Aut(\mathcal{C}) = Aut_I(\mathcal{C})$.
2. If $R(x, f(x), y)$ and $R(y, f^{-1}(x), x)$ then there is no submodel M_0 such that $t(x/M_0) = t(y/M_0)$. So, there is no $g \in \Delta$ such that $y = g(x)$.
3. If $k < n/2$, $R(x, f(x), y)$ and $R(y, f^{-1}(x), x)$ then there is no $g \in \Delta^k$ such that $y = g(x)$.

It remains to glue all these theories together.

Some Problems

1. If T is simple, we have seen that the notion of invariant strong type and compact strong type coincide. But can we find an example where this is not the notion of Shelah strong type?

Partial answers:

YES (Hrushovski) but with a Robinson theory.

NO, (Buechler), if T is low.

2. What about the group $Autf_K(\mathcal{C})/Autf_I(\mathcal{C})$. This group is naturally isomorphic to H, where H is the topological closure of $\{e\}$ in $Gal_I(T)$. We have an example of a countable theory where this group has cardinality $2^{\aleph_0}$. Are there other constrains? Can this group be finite (but not trivial), or countable. The only thing that we know is that, for a countable theory, if $\|Autf_K(\mathcal{C})/Autf_I(\mathcal{C})\| > \aleph_0$ then $\|Autf_K(\mathcal{C})/Autf_I(\mathcal{C})\| = 2^{\aleph_0}$.

If $Aut_I(\mathcal{C})$ is not closed in $Aut(\mathcal{C})$ (endowed with the pointwise topology), then certainly $Autf_K(\mathcal{C})/Autf_I(\mathcal{C})$ cannot be countable.