THE FIELDS INSTITUTE
FOR RESEARCH IN MATHEMATICAL SCIENCES

ABSTRACTS 1.2

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Some results and questions in mass optimization problems

We consider the optimization problem

$$\max\{\mathcal{E}(f,\Sigma,\mu)\ :\ \mu\in\mathcal{M}^+(m,\overline{\Omega})\}$$

where $\mathcal{E}(f,\Sigma,\mu)$ is the energy associated to μ:

$$\mathcal{E}(f,\Sigma,\mu)=\inf\{\frac{1}{2}\int|Du|^2\,d\mu-\langle f,u\rangle\ :\ u\in\mathcal{D}(\mathbf{R}^n),\ u=0\text{ on }\Sigma\}.$$

The datum f is a signed measure with finite total variation (and zero average if Σ is empty), while Ω is an open subset of $\mathbf{R}^n$, Σ is a closed subset of $\overline{\Omega}$, and $\mathcal{M}^+(m,\overline{\Omega})$ is the class of nonnegative measures on $\overline{\Omega}$ with mass m. We show that the optimization problem above admits a solution which is a measure, not in $L^1(\Omega)$ in general. This solution comes out by solving a mass transportation problem, for a suitable distance $d_{\Omega,\Sigma}$ and with the associated Monge-Kantorovich PDE equation of the form

$$\begin{cases}-\operatorname{div}(\mu D_\mu w)=f & \text{on }\mathbf{R}^n\setminus\Sigma\\ w\in\operatorname{Lip}_1(\Omega,\Sigma), & |D_\mu w|=1\ \mu\text{-a.e.},\quad \mu(\Sigma)=0\end{cases}$$

where $\operatorname{Lip}_1(\Omega,\Sigma)$ is the class of all Lipschitz functions on Ω with constant 1, which vanish on Σ.

We also address some problems occurring in the optimization problems which arise when we consider the Dirichlet region Σ as an unknown.