Introduction to Shimura Varieties

J.S. Milne

Preliminary version. June 2003*

Abstract

This is an introduction to the theory of Shimura varieties, or, in other words, to the arithmetic theory of automorphic functions and holomorphic automorphic forms. (June 3, §1,2; June 6, §3,4.)

Introduction

The arithmetic properties of elliptic modular functions and forms were extensively studied in the 1800s, culminating in the beautiful Kronecker Jugendtraum. Hilbert emphasized the importance of extending this theory to functions of several variables in the twelfth of famous problems at the International Congress in 1900. The first tentative steps in this direction were taken by Hilbert himself and his students Blumenthal and Hecke in their study of what are now called Hilbert (or Hilbert-Blumenthal) modular varieties. As the theory of complex functions of several variables matured, other quotients of bounded symmetric domains by arithmetic groups were studied (Siegel, Braun, and others). However, the modern theory of Shimura varieties[1] only really began with the development of the theory of abelian varieties with complex multiplication by Shimura, Taniyama, and Weil in the mid-1950s, and with the subsequent proof by Shimura of the existence of canonical models for certain families of Shimura varieties. In two fundamental articles, Deligne recast the theory in the language of abstract reductive groups and extended Shimura's results on canonical models. Langlands made Shimura varieties a central part of his program, both as a source of representations of Galois groups and as tests for his conjecture that all motivic L-functions are automorphic.

These notes are an introduction to the theory of Shimura varieties from the perspective of Deligne and Langlands. Because of their brevity, I have not been able to include proofs, although I have tried to explain why statements are true. Headings "PROOF" should be understood in this spirit.

Throughout the text, references are usually to the most accessible source for a statement or proof, not the original source.

Prerequisites

Beyond the mathematics that students usually acquire by the end of their first year of graduate work (a little complex analysis, topology, algebra, differential geometry,...), I assume some familiarity with algebraic number theory (e.g., §§1–4, 7 of Milne ANT), algebraic geometry (e.g., §§1–5, 9, 13 of Milne AG), algebraic groups (e.g., Murnaghan 2003), and elliptic modular curves (e.g., §§1–8 of Milne MF and Milne SC).

Notations and conventions

Unless indicated otherwise, vector spaces are assumed to be finite dimensional and free $\mathbb{Z}$-modules of finite rank. The dual $\mathrm{Hom}(V,k)$ of a vector space V is denoted $V^\vee$.

A superscript $^+$ (resp. $^\circ$) denotes a(n identity) connected component relative to a euclidean topology (resp. a Zariski topology). For example, if $G = \mathrm{GL}_2$, then $G^\circ = G$ and $G(\mathbb{R})^+$ consists of the matrices with $\det > 0$.

References

Akhiezer, D. N. 1990, Homogeneous complex manifolds, *in* 'Several Complex Variables IV', Springer Verlag, Berlin Heidelberg New York, pp. 195–244.

Baily, Jr., W. L. & Borel, A. 1966, 'Compactification of arithmetic quotients of bounded symmetric domains', *Ann. of Math. (2)* **84**, 442–528.

Boothby, W. M. 1975, *An introduction to differentiable manifolds and Riemannian geometry*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London.

[1]The term ***Shimura variety*** was introduced by Langlands (1977), although earlier ***Shimura curve*** had been used for the varieties of dimension one (Ihara 1968).

Borel, A. 1966, Reduction theory for arithmetic groups, *in* 'Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)', Amer. Math. Soc., Providence, R.I., pp. 20–25.

Borel, A. 1969, *Linear algebraic groups*, Notes taken by Hyman Bass, W. A. Benjamin, Inc., New York-Amsterdam.

Bourbaki, N. 1981, *Groupes et Algèbres de Lie. Chapitres 4,5 et 6*, Elements of Mathematics, Masson, Paris.

Bourbaki, N. 1989, *General topology. Chapters 1–4*, Elements of Mathematics, Springer-Verlag, Berlin.

Deligne, P. 1971*a*, 'Travaux de Shimura'. Preliminary version of Deligne 1971b.

Deligne, P. 1971*b*, Travaux de Shimura, *in* 'Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389', Springer, Berlin, pp. 123–165. Lecture Notes in Math., Vol. 244.

Deligne, P. 1973, *Espaces hermitiens symétriques, Spring 1973*, Transcription available at www.jmilne.org/math/.

Deligne, P. 1979, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, *in* 'Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2', Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., pp. 247–289.

Deligne, P. 1982, Hodge cycles on abelian varieties. (Notes by J.S. Milne), *in* 'Hodge cycles, motives, and Shimura varieties', Vol. 900 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, pp. 9–100.

Greenberg, M. J. 1967, *Lectures on algebraic topology*, W. A. Benjamin, Inc., New York-Amsterdam.

Harris, J. 1992, *Algebraic geometry*, Vol. 133 of *Graduate Texts in Mathematics*, Springer-Verlag, New York.

Hartshorne, R. 1977, *Algebraic geometry*, Springer-Verlag, New York.

Helgason, S. 1978, *Differential geometry, Lie groups, and symmetric spaces*, Vol. 80 of *Pure and Applied Mathematics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York.

Humphreys, J. E. 1972, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York.

Humphreys, J. E. 1975, *Linear algebraic groups*, Springer-Verlag, New York.

Humphreys, J. E. 1978, *Introduction to Lie algebras and representation theory*, Vol. 9 of *Graduate Texts in Mathematics*, Springer-Verlag, New York.

Ihara, Y. 1968, *On congruence monodromy problems. Vol. 1*, Lecture Notes, No. 1, Department of Mathematics, University of Tokyo, Tokyo.

Iitaka, S. 1982, *Algebraic geometry*, Vol. 76 of *Graduate Texts in Mathematics*, Springer-Verlag, New York.

Jannsen, U. 1992, ‘Motives, numerical equivalence, and semi-simplicity’, *Invent. Math.* **107**(3), 447–452.

Kelley, J. L. 1955, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York-London.

Kobayashi, S. 1972, *Transformation groups in differential geometry*, Springer-Verlag, New York.

Kottwitz, R. E. 1984, ‘Shimura varieties and twisted orbital integrals’, *Math. Ann.* **269**(3), 287–300.

Kottwitz, R. E. 1992, ‘Points on some Shimura varieties over finite fields’, *J. Amer. Math. Soc.* **5**(2), 373–444.

Krantz, S. G. 1992, *Function theory of several complex variables*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA.

Kwack, M. H. 1969, ‘Generalization of the big Picard theorem’, *Ann. of Math. (2)* **90**, 9–22.

Langlands, R. P. 1977, ‘Shimura varieties and the Selberg trace formula’, *Canad. J. Math.* **29**(6), 1292–1299.

Margulis, G. A. 1991, *Discrete subgroups of semisimple Lie groups*, Vol. 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, Springer-Verlag, Berlin.

Milne, J. S. 1986, Abelian varieties, *in* ‘Arithmetic geometry (Storrs, Conn., 1984)’, Springer, New York, pp. 103–150.

Milne, J. S. AG, *Algebraic Geometry 1998*, www.jmilne.org/math/.

Milne, J. S. ANT, *Algebraic Number Theory 1998*, www.jmilne.org/math/.

Milne, J. S. CFT, *Class Field Theory 1997*, www.jmilne.org/math/.

Milne, J. S. FT, *Fields and Galois Theory 2002*, www.jmilne.org/math/.

Milne, J. S. MF, *Modular Functions and Modular Forms 1997*, www.jmilne.org/math/.

Milne, J. S. SC, *Canonical Models of Shimura Curves, April 4, 2003*, Available at www.jmilne.org/math/.

Mumford, D. 1977, ‘Hirzebruch’s proportionality theorem in the noncompact case’, *Invent. Math.* **42**, 239–272.

Mumford, D. 1999, *The red book of varieties and schemes*, Vol. 1358 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

Murnaghan, F. 2003, ‘Background from algebraic groups’. Lectures at the Fields Institute, June 2–6, 2003.

Platonov, V. & Rapinchuk, A. 1994, *Algebraic groups and number theory*, Vol. 139 of *Pure and Applied Mathematics*, Academic Press Inc., Boston, MA.

Serre, J.-P. 1964, ‘Sur les groupes de congruence des variétés abéliennes’, *Izv. Akad. Nauk SSSR Ser. Mat.* **28**, 3–20.

Serre, J.-P. 1968, *Abelian l-adic representations and elliptic curves*, McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute, W. A. Benjamin, Inc., New York-Amsterdam.

Serre, J.-P. 1987, *Complex semisimple Lie algebras*, Springer-Verlag, New York.

Shafarevich, I. R. 1994, *Basic algebraic geometry. 1,2*, Springer-Verlag, Berlin.

Shimura, G. 1963, 'On analytic families of polarized abelian varieties and automorphic functions', *Ann. of Math. (2)* **78**, 149–192.

Springer, T. A. 1998, *Linear algebraic groups*, Vol. 9 of *Progress in Mathematics*, Birkhäuser Boston Inc., Boston, MA.

Vasiu, A. 1999, 'Integral canonical models of Shimura varieties of preabelian type', *Asian J. Math.* **3**(2), 401–518.

Voisin, C. 2002, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge University Press.

Weil, A. 1956, 'The field of definition of a variety', *Amer. J. Math.* **78**, 509–524.

Wells, R. O. 1980, *Differential analysis on complex manifolds*, Vol. 65 of *Graduate Texts in Mathematics*, Springer-Verlag, New York.

Witte, D. 2001, *Introduction to Arithmetic Groups*, arXiv:math.DG/0106063.

Wolf, J. A. 1984, *Spaces of constant curvature*, Publish or Perish Inc., Houston, TX.

1 Hermitian symmetric domains

In this section, I describe the complex manifolds that play the role in higher dimensions of the complex upper half plane, or, equivalently, the open unit disk:

$$\{z \in \mathbb{C} \mid \Im(z) > 0\} = \mathcal{H}_1 \underset{-i\frac{z+1}{z-1} \leftarrow z}{\overset{z \mapsto \frac{z+i}{z-i}}{\rightleftarrows}} \mathcal{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$$

Review of real manifolds

A ***manifold*** M ***of dimension*** n is a Hausdorff topological space that is locally isomorphic to $\mathbb{R}^n$ and admits a countable basis of open subsets . A ***chart*** of M is a homeomorphism φ from an open subset U of M onto an open subset of $\mathbb{R}^n$.

Smooth manifolds

I use the terms differentiable, smooth, and C^∞ interchangeably. A ***smooth manifold*** is a manifold M endowed with a ***smooth structure***, i.e., a sheaf $\mathcal{O}_M$ of $\mathbb{R}$-valued functions such that $(M, \mathcal{O}_M)$ is locally isomorphic to $\mathbb{R}^n$ endowed with its sheaf of differentiable functions. A smooth structure on a manifold M can be defined by a family $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ of charts such that $M = \bigcup U_\alpha$ and

$$\varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth for all α, β.

Let $(M, \mathcal{O}_M)$ be a smooth manifold, and let $\mathcal{O}_{M,p}$ denote the ring of germs of smooth functions at p. The ***tangent space*** T_pM at p is the $\mathbb{R}$-vector space of derivations $X_p \colon \mathcal{O}_{M,p} \to \mathbb{R}$. A ***smooth vector field*** on an open subset of U of M is a family $X = (X_p)_{p \in U}$, $X_p \in T_p(M)$, such that, for any differentiable function f on an open subset of U, $p \mapsto Xf(p) =_{\text{df}} X_pf$ is differentiable. A ***smooth*** r***-tensor field*** on an open subset U of M is a family $f = (f_p)_{p \in M}$ of multilinear mappings $f_p \colon T_pM \times \cdots \times T_pM \to \mathbb{R}$ (r copies of T_pM) such that, for any smooth vector fields $X_1, \ldots, X_r$ on an open subset of U, $p \mapsto f_p(X_1, \ldots, X_r)$ is a smooth function. A 1-tensor field is also called a ***covector field***. A ***smooth*** (r, s)***-tensor field*** is a family $f_p \colon (T_pM)^r \times (T_pM)^{\vee s} \to \mathbb{R}$ satisfying the obvious condition. Note that a smooth $(1,1)$-field can be identified with a family of endomorphisms $f_p \colon T_pM \to T_pM$ such that, for any smooth vector field X, $p \mapsto f_p(X_p)$ is also smooth.

A ***riemannian metric on*** M is a 2-tensor field g such that, for all $p \in M$, g_p is symmetric and positive definite.

Real-analytic manifolds

To define a ***real-analytic manifold***, simply replace "smooth" with "real-analytic" in the above definition. The elementary theory of real-analytic manifolds is very similar to that of smooth manifolds except that one must remember that there are many fewer real-analytic functions. For example, not every germ of a real-analytic function is represented by a real-analytic function on the whole of M. Thus, one must state things locally, as we did in the preceding subsubsection.

Review of hermitian forms

To give a complex vector space V amounts to giving a real vector space V together with an endomorphism $J\colon V \to V$ such $J^2 = -1$. Then $z = a + bi$ acts as $a + bJ$. A ***hermitian form*** on (V, J) is a mapping $(\,|\,)\colon V \times V \to \mathbb{C}$ that is linear in the first variable, semilinear in the second, and satisfies $(v|u) = \overline{(u|v)}$. When we write

$$(u|v) = \varphi(u,v) - i\psi(u,v), \quad \varphi(u,v), \psi(u,v) \in \mathbb{R}, \tag{1}$$

then φ and ψ are $\mathbb{R}$-bilinear, and

$$\varphi \text{ is symmetric} \qquad \varphi(Ju, Jv) = \varphi(u,v), \tag{2}$$

$$\psi \text{ is skew-symmetric} \qquad \psi(Ju, Jv) = \psi(u,v), \tag{3}$$

$$\psi(u,v) = -\varphi(u, Jv), \qquad \varphi(u,v) = \psi(u, Jv). \tag{4}$$

As $(u|u) = \varphi(u,u)$, $(\,|\,)$ is positive definite if and only if φ is positive definite. Conversely, if φ satisfies (2) (resp. ψ satisfies (3)), then the formulas (4) and (1) define a hermitian form:

$$(u|v) = \varphi(u,v) + i\varphi(u, Jv) \quad (\text{resp. } (u|v) = \psi(u, Jv) - i\psi(u,v)) \tag{5}$$

Review of real algebraic groups

PROPOSITION 1.1. *A real Lie group is algebraic if and only if it has a faithful representation on a finite dimensional vector space.*

Hence, adjoint Lie groups (i.e., centreless groups) are algebraic, because the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Lie}\, G$ is faithful.

Let G be a connected algebraic group over $\mathbb{R}$, and let $g \mapsto \overline{g}$ denote complex conjugation on $G(\mathbb{C})$. An involution θ of G defines a real form $G^{(\theta)}$, which is characterized by the fact that complex multiplication on $G^{(\theta)}(\mathbb{C}) = G(\mathbb{C})$ is $g \mapsto \theta(\overline{g})$. A ***Cartan involution*** is an involution such that $G^{(\theta)}(\mathbb{R})$ is compact, i.e., θ is a Cartan if $\{g \in G(\mathbb{C}) \mid g = \theta(\overline{g})\}$ is compact.

EXAMPLE 1.2. Let $G = \mathrm{SL}_2$, and let $\theta = \mathrm{ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Thus, $\mathrm{SL}_2^{(\theta)}(\mathbb{R})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{C})$ such that $a = \overline{d}$, $c = -\overline{b}$, i.e., the complex matrices $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. This is certainly compact, and so θ is a Cartan involution for SL_2.

THEOREM 1.3. *There exists a Cartan involution if and only if G is reductive, in which case any two are conjugate by an element of $G(\mathbb{R})$.*

EXAMPLE 1.4. (a) Let $G = \mathrm{GL}(V)$ with V a real vector space. The choice of a basis for V determines a transpose operator $M \mapsto M^t$, and $M \mapsto (M^t)^{-1}$ is obviously a Cartan involution. The theorem says that all Cartan involutions arise this way.

(b) Let G be a connected algebraic group over $\mathbb{R}$, and let $G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation of G. Then G is reductive if and only if G is stable under $g \mapsto g^t$ for a suitable choice of a basis for V, in which case the restriction of $g \mapsto (g^t)^{-1}$ is a Cartan involution; all Cartan involutions of G arise in this way (Satake 1980, I 4.4).

(c) If G is simple, then $G_{\mathbb{C}}$ is either simple or else it decomposes as the product of two conjugate groups $G_1 \times G_2$. In the second case, $G \approx \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_1$ (restriction of scalars, see Springer 1998, 11.4.16) and $G(\mathbb{R}) \approx G_1(\mathbb{C})$, which is not compact.[2] Therefore, if G simple and compact, then $G_{\mathbb{C}}$ is also simple.

PROPOSITION 1.5. *Let G be a connected algebraic group over $\mathbb{R}$. If $G(\mathbb{R})$ is compact, then every real representation of $G \to \mathrm{GL}(V)$ carries a G-invariant positive definite symmetric bilinear form; conversely, if one faithful real representation of G carries such a form, then $G(\mathbb{R})$ is compact.*

PROOF. Let $\rho\colon G \to \mathrm{GL}(V)$ be a real representation of G. If $G(\mathbb{R})$ is compact, then its image H in $\mathrm{GL}(V)$ is compact. Let dh be the Haar measure on H, and choose a positive definite symmetric bilinear form $\langle\,|\,\rangle$ on V. Then the form

$$\langle u|v\rangle' = \int_H \langle hu|hv\rangle dh$$

is G-invariant, and it is still symmetric, positive, and bilinear. For the converse, choose an orthonormal basis for the form. Then $G(\mathbb{R})$ becomes identified with the set of real matrices A such that $A^t \cdot A = I$, which is closed and bounded. □

REMARK 1.6. The proposition can be restated for complex representations: if $G(\mathbb{R})$ is compact then every finite-dimensional complex representation of G carries a G-invariant positive definite Hermitian form; conversely, if some faithful real representation of G carries a G-invariant positive definite Hermitian form, then $G(\mathbb{R})$ is compact.

Let G be a real algebraic group, and let C be an element of $G(\mathbb{R})$ whose square is central. A ***C-polarization*** on a representation V of G is a G-invariant bilinear form φ such that $\varphi(u, Cv)$ is symmetric and positive definite.

PROPOSITION 1.7. *If $\mathrm{ad}C$ is a Cartan involution of G, then every real representation of G carries a C-polarization; conversely, if one faithful representation of G carries a C-polarization, then $\mathrm{ad}C$ is a Cartan involution.*

PROOF. Let $G \to \mathrm{GL}(V)$ be representation of G, and let φ be a G-invariant bilinear form on V. Define

$$\varphi'\colon V(\mathbb{C}) \times V(\mathbb{C}) \to \mathbb{C}, \quad \varphi'(u,v) = \varphi_{\mathbb{C}}(u, \overline{v}).$$

Then φ' is sesquilinear, and for all $g \in G(\mathbb{C})$,

$$\varphi'(gu, \overline{g}v) = \varphi_{\mathbb{C}}(gu, g\overline{v}) = \varphi_{\mathbb{C}}(u, \overline{v}) = \varphi'(u,v), \quad u, v \in V(\mathbb{C}).$$

On replacing v with Cv, we find that

$$\varphi'(gu, CC^{-1}\overline{g}Cv) = \varphi'(u, Cv), \quad g \in G(\mathbb{C}), \quad u, v \in V(\mathbb{C}). \tag{6}$$

Let $\varphi_C(u,v) = \varphi(u, Cv)$ and let φ'_C be the associated sesquilinear form on $V(\mathbb{C})$. Then (6) states that φ'_C is $G^{(\theta)}(\mathbb{C})$-invariant, where $\theta = \mathrm{ad}(C)$, and so φ_C is $G^{(\theta)}(\mathbb{R})$-invariant. Therefore, if the representation is faithful and φ_C is positive definite, then $G^{(\theta)}(\mathbb{R})$ is compact (1.5).

Conversely, if $G^{(\theta)}$ is compact, then every representation $G \to \mathrm{GL}(V)$ carries a $G^{(\theta)}$-invariant positive definite form ψ (1.5), and the above argument can be run backwards to show that $(u,v) \mapsto \psi(u, C^{-1}v)$ is a C-polarization of V. □

[2] If $G_1 \neq 1$, it contains a torus $T \neq 1$, and $T(\mathbb{C}) \approx \mathbb{C}^{\times r}$ for some $r \geq 1$.

Complex manifolds

A ***complex manifold*** is a manifold M endowed with a ***complex structure***, i.e., a sheaf $\mathcal{O}_M$ of $\mathbb{C}$-valued functions such that $(M, \mathcal{O}_M)$ is locally isomorphic to $\mathbb{C}^n$ with its sheaf of complex-analytic functions. A complex structure on a manifold M can be defined by a family $\varphi_\alpha \colon U_\alpha \to \mathbb{C}^n$ of charts such that $M = \bigcup U_\alpha$ and

$$\varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is complex-analytic for all α, β. Such a family also defines a smooth structure on M and a real-analytic structure. Thus, a complex manifold M has an underlying smooth structure $\mathcal{O}_M^\infty$ and an underlying real-analytic structure $\mathcal{O}_M^{\mathrm{an}}$. A ***tangent vector*** at a point p of a complex manifold is a derivation $\mathcal{O}_{M,p} \to \mathbb{C}$. The tangent spaces T_pM (M as a complex manifold) and T_pM^∞ (M as a smooth manifold) can be identified. Thus, a complex structure on a smooth (or real-analytic) manifold M provides each tangent space T_pM with a complex structure. Explicitly, complex local coordinates $z^1, \ldots, z^n$ at a point p of M define real local coordinates $x^1, \ldots, x^n, y^1, \ldots, y^n$ with $z^r = x^r + iy^r$. The real and complex tangent spaces have bases $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$ and $\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}$ respectively. Under the natural identification of the two spaces, $\frac{\partial}{\partial z^r} = \frac{\partial}{\partial x^r} + i\frac{\partial}{\partial y^r}$.

Recall that a smooth function $f \colon \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it satisfies the Cauchy-Riemann condition. This last condition has a geometric interpretation: it requires that $df_p \colon T_p\mathbb{C} \to T_{f(p)}\mathbb{C}$ be $\mathbb{C}$-linear for all $p \in \mathbb{C}$.[3] This allows us to define a smooth function $f \colon \mathbb{C}^r \to \mathbb{C}$ to be ***holomorphic*** if the maps $df_z \colon T_z\mathbb{C}^r \to T_{f(z)}\mathbb{C}$ are $\mathbb{C}$-linear for all z. Just as in the one-variable case, a smooth function is holomorphic if and only if it is complex-analytic (Voisin 2002, 1.17). From now on, I shall say holomorphic for complex-analytic, and reserve analytic to mean real-analytic.

An ***almost-complex structure*** on a smooth (resp. analytic) manifold M is a smooth (resp. analytic) tensor field $(J_p)_{p \in M}$, $J_p \colon T_pM \to T_pM$, such that $J_p^2 = -1$ for all p. A complex structure on smooth manifold endows it with an almost-complex structure. In terms of complex local coordinates $z^1, \ldots, z^n$ in a neighbourhood of a point p on a complex manifold and the corresponding real local coordinates $x^1, \ldots y^n$, J_p acts by

$$\frac{\partial}{\partial x^k} \mapsto \frac{\partial}{\partial y^k}, \quad \frac{\partial}{\partial y^k} \mapsto -\frac{\partial}{\partial x^k}. \tag{7}$$

The functor from complex manifolds to almost-complex manifolds is fully faithful, i.e., a smooth map $\alpha \colon M \to N$ of complex manifolds such that the maps $d\alpha_p$ are $\mathbb{C}$-linear is holomorphic. It is known that an almost-complex structure on a smooth (or complex) manifold arises from a complex structure if and only if

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY] \tag{8}$$

[3] Let $z = x + iy$, so that $dz = dx + idy$. Let $f = u + iv$ be a smooth function $\mathbb{C} \to \mathbb{C}$. Then

$$df = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right),$$

which equals $(a + bi)dz$ if and only if $a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $b = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Thus, $(df)_p$ is $\mathbb{C}$-linear if and only if f satisfies the Cauchy-Riemann equations at p, in which case $(df)_p = (\frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p))(dz)_p$.

(see Voisin 2002, 2.2.3, for the analytic case). A more elementary condition is that an almost complex structure J on a smooth manifold M covered by local coordinate neighbourhoods on which J takes the form (7) arises from a complex structure (because the condition forces the coordinate changes to be holomorphic).

From these statements, we see that a complex manifold can be regarded as a smooth manifold endowed with complex structures on its tangent spaces varying smoothly and satisfying (8). Moreover, a hermitian metric on M can be regarded as a riemannian metric g such that $g_p(JX, JY) = g_p(X, Y)$ for all $p \in M$ and $X, Y \in T_pM$.

Hermitian symmetric spaces

DEFINITION 1.8. (a) A manifold M (smooth, riemannian, complex, ...) is ***homogeneous*** if for any pair of points (p_1, p_2) of M, there exists an automorphism α of M such that $\alpha(p_1) = p_2$; in other words, the group $\mathrm{Aut}(M)$ acts transitively on M.

(b) A ***symmetry at a point*** p of a manifold M is an involution $s_p\colon M \to M$ having p as an isolated fixed point.

(c) A manifold is ***symmetric*** if it is homogeneous and admits a symmetry at one (hence every) point.

ASIDE 1.9. A riemannian manifold (M, g) that admits a symmetry at each point is homogeneous, and hence symmetric. The symmetry s_p at p acts as -1 on T_pM (see 1.15 below). Because s_p is an isometry, it maps a point P on a geodesic γ through x to the point $P' \neq P$ on γ equidistant from x.

DEFINITION 1.10. A ***hermitian symmetric space*** is a symmetric hermitian manifold (equivalently, a symmetric riemannian almost complex manifold (M, g) satifying (8) and such that $g_p(JX, JY) = g_p(X, Y)$).

EXAMPLE 1.11. (a) Let $\mathcal{H}_1$ be the complex upper half plane. Then $\mathrm{SL}_2(\mathbb{R})$ acts on M by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$. For any $z = x + iy \in M$, $z = \left(\begin{smallmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{smallmatrix}\right) i$, and so $\mathcal{H}_1$ is homogeneous. The isomorphism $z \mapsto -1/z$ is a symmetry at $i \in M$, and so M is symmetric. The metric $dxdy/y^2$, which is invariant under the action of $\mathrm{Hol}(H) = \mathrm{PSL}_2(\mathbb{R})$, has the hermitian property (3).

(b) The projective line $\mathbb{P}^1(\mathbb{C}) \cong \{\mathrm{sphere}\}$ is a hermitian symmetric space. The group of rotations is transitive, and reflection along a geodesic (great circle) is a symmetry. The restriction to the sphere of the euclidean metric on $\mathbb{R}^3$ gives a hermitian metric on $\mathbb{P}^1(\mathbb{C})$.

(c) The complex line (plane) $\mathbb{C}$ is a hermitian symmetric space, as is any quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a lattice. The group of translations is transitive, and $z \mapsto -z$ is a symmetry at 0. The euclidean metric is hermitian.

Curvature. Recall that, for a plane curve, the curvature at a point p is $1/r$ where r is the radius of the circle that best approximates the curve at p. For a space surface, the principal curvatures at a point p are the maximum and minimum of the signed curvatures of the curves obtained by cutting the surface with a plane through the normal at p (the sign is positive or negative according as the curve bends towards the normal or away). Although the principal curvatures depend on the embedding into $\mathbb{R}^3$, their product (the ***sectional***, or ***Gauss, curvature*** at p) does not (Gauss's Theorema Egregium) and is well-defined for any riemannian surface. More generally, for a point p on any riemanian manifold M, one can define the ***sectional curvature*** $K(p, E)$ of the submanifold cut out by the geodesics tangent to a 2-dimensional subspace E of T_pM. Intuitively, positive curvature means that the geodesics through a point converge, and negative curvature means that they diverge. The geodesics in the upper half plane are the half-lines and semicircles orthogonal to the real axis. Clearly, they diverge — in fact, this is Poincaré famous model of noneuclidean geometry in which given a "line" and a point not on it, there are more than one "lines" through the point not meeting the first. More prosaically, one can compute that the sectional curvature is -1. The Gauss curvature of $\mathbb{P}^1(\mathbb{C})$ is positive, and that of $\mathbb{C}/\Lambda$ is zero.

The three types of hermitian symmetric spaces

Recall that the group $\mathrm{Is}(M, g)$ of isometries of riemannian manifold (M, g) is a Lie group for the compact-open topology (Theorem of Steenrod and Myers; for the proof, Kobayashi 1972, II.1). The group of automorphisms of hermitian symmetric space is a closed subgroup of the isometries of the underlying riemannian manifold, and hence is a Lie group. Every symmetric riemannian manifold, hence every hermitian symmetric space is complete.

Name	Example	simply connected	curvature	$\mathrm{Aut}(X)$
noncompact type	$\mathcal{H}_1$	yes	negative	adjoint, noncompact
compact type	$\mathbb{P}^1(\mathbb{C})$	yes	positive	adjoint, compact
Euclidean	$\mathbb{C}/\Lambda$	not necessarily	zero	

An adjoint group is a semisimple algebraic group with trivial centre. Every hermitian symmetric space is a product $X^0 \times X^- \times X^+$ with X^0 Euclidean, X^- of noncompact type, and X^+ of compact type. See Helgason 1978 or Wolf 1984.

DEFINITION 1.12. A ***hermitian symmetric domain*** is a hermitian symmetric space of noncompact type.

EXAMPLE 1.13 (SIEGEL UPPER HALF SPACE). The ***Siegel upper half space*** $\mathcal{H}_g$ ***of degree*** g consists of the symmetric complex $g \times g$ matrices with positive definite imaginary part, i.e.,

$$\mathcal{H}_g = \{X + iY \in M_g(\mathbb{C}) \mid X = X^t, \quad Y > 0\}.$$

Note that the map $Z = (z_{ij}) \mapsto (z_{ij})_{i \geq j}$ identifies $\mathcal{H}_g$ with an open subset of $\mathbb{C}^{g(g+1)/2}$. Let $\mathrm{Sp}_{2g}(\mathbb{R})$ be the group fixing the skew-symmetric form $\sum_{i=1}^{g} x_i y_{-i} - \sum_{i=1}^{g} x_{-i} y_i$. Then

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{ll} A^tC = C^tA & A^tD - C^tB = I \\ D^tA - B^tC = I & B^tD = D^tB \end{array} \right\},$$

and $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on $\mathcal{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

The element $J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ acts as an involution on $\mathcal{H}_g$, and has iI_g as its only fixed point.

Example: Bounded symmetric domains.

A domain in $\mathbb{C}^n$ is a nonempty open connected subset. It is symmetric if it is symmetric as a complex manifold. For example, $\mathcal{H}_1$ is a symmetric domain, and $\mathcal{D}_1$ is a bounded symmetric domain.

PROPOSITION 1.14. *Every bounded domain has canonical hermitian metric (called the Bergman(n) metric). Moreover, this metric has negative curvature.*

PROOF. Initially, let D be any domain in $\mathbb{C}^n$. Then the set of square-integrable measurable functions $f\colon D \to \mathbb{C}$ forms a complete separable Hilbert space with inner product $\langle f, g\rangle = \int_D f\overline{g}dv$. Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{O}^2(D)$, and set $K(z) = \sum e_i(z)\cdot\overline{e_i(z)}$. Then $K(z)$ is a real analytic function on D, independent of the choice of basis. If D is bounded, then all polynomial functions on D are square-integrable, and so certainly $K(z) > 0$ for all z. Hence, $\log(K(z))$ is real analytic and the equations

$$h = \sum h_{ij}dz^i d\overline{z}^j, \quad h_{ij}(z) = \frac{\partial^2}{\partial z^i \partial \overline{z}^j}K(z),$$

define a hermitian metric on D — see Krantz 1992 or Helgason 1978, VIII 3. □

The Bergman metric, being truly canonical, is invariant under the action of automorphisms of D. Hence, a bounded symmetric domain is a hermitian symmetric domain. In the table 11 below, we list some examples. Conversely, Harish-Chandra has shown that every hermitian symmetric domain can be embedded into some $\mathbb{C}^n$ as a bounded symmetric domain. In particular, it has a canonical (Bergman) hermitian metric.

The homomorphism $u_p\colon U_1 \to Hol(D)$

Initially, let (M, g) be a riemannian manifold such that, for all $p \in M$, there is a symmetry at p.

PROPOSITION 1.15. *A symmetry s_p at p acts as -1 on T_pM.*

PROOF. Because $s_p^2 = 1$, $(ds_p)^2 = 1$, and so ds_p acts semisimply on T_pM with eigenvalues ± 1. Suppose $+1$ occurs, and let X be a tangent vector with eigenvector $+1$. There is a unique geodesic $\gamma\colon I \to M$ with

$$\gamma(0) = p, \dot{\gamma}(t) = X. \tag{9}$$

Because s_p is an isometry, $s_p \circ \gamma$ is also a geodesic, and it satisfies (9). Therefore, $\gamma = s_p \circ \gamma$ and p is not an isolated fixed point of s_p. □

By a ***canonical tensor*** on (M, g) I mean any tensor canonically derived from g, and hence fixed by any isometry of (M, g). For example, the riemannian connection ∇ is canonical.

PROPOSITION 1.16. *On (M,g) every canonical r-tensor with r odd is zero. In particular, parallel translation of 2-dimensional subspaces does not change the sectional curvature.*

PROOF. Let t be a canonical r-tensor. Then

$$t_p = t_p \circ (ds_p)^r \overset{1.15}{=} (-1)^r t_p,$$

and so $t=0$ if r is odd. For the second statement, one only has to observe that the derivative relative to a tangent vector of the plane spanned by a pair of tangent vectors is a 3-tensor. □

PROPOSITION 1.17. *Let (M,g) and (M',g') be riemannian manifolds in which parallel translation of 2-dimensional subspaces does not change the sectional curvature. Let $a\colon T_pM \to T_{p'}M'$ be a linear isometry such that $K(p,E)=K(p',aE)$ for every 2-dimensional subspace $E\subset T_pM$. Then $\exp_p(X)\mapsto \exp_{p'}(aX)$ is an isometry of a normal neighbourhood of p onto a normal neighbourhood of p'.*

PROOF. This follows from comparing the expansions of the riemann metrics in terms of normal geodesic coordinates. See Wolf 1984, 2.3.6. □

REMARK 1.18. If in (1.17) M and M' are complete and simply connected, then $\exp_p(X)\to \exp_{p'}(aX)$ extends to an isometry of $M\to M'$.

THEOREM 1.19. *Let D be a hermitian symmetric space, and let $G=\mathrm{Aut}(D)$. For each $p\in M$, there exists a unique homomorphism $u_p\colon U_1\to G$ such that $u_p(z)$ acts on T_pM as multiplication by z.*

PROOF. Each z with $|z|=1$ defines an automorphism of (T_pD,h_p), and one checks that it preserves sectional curvatures. According to (1.16, 1.17, 1.18), there exists an isometry $u_p(z)\colon D\to D$ such that $du_p(z)_p$ is multiplication by z. It is holomorphic because it is $\mathbb{C}$-linear on the tangent spaces. □

EXAMPLE 1.20. For $\mathcal{H}_1$ the upper half plane and $p=i$, let $h_p\colon \mathbb{C}^\times \to \mathrm{SL}_2(\mathbb{R})$ be the homomorphism $z=a+ib\mapsto \left(\begin{smallmatrix} a & b\\ -b & a\end{smallmatrix}\right)$. Then $h_p(z)$ acts on $T_p\mathcal{H}_1$ as multiplication by $z/\overline{z}$ (because $\frac{d}{dz}\left(\frac{az+b}{-bz+a}\right)|_i = \frac{a^2+b^2}{(a-bi)^2}$). Note that $h_p(-1)=1$. For $z\in U_1$, choose a square root $\sqrt{z}$, and set $u_p(z)=h_p(\sqrt{z}) \mod \pm I$. Then u_p is a well-defined homomorphism $U_1\to \mathrm{PSL}_2(\mathbb{R})$, and $du_p(z)_p=z$.

Classification of hermitian symmetric domains in terms of real groups

A representation $\rho\colon U_1\to \mathrm{GL}(V)$ of U_1 on a complex vector space V decomposes it into a direct sum $V=\oplus V^n$ where $V^n=\{v\in V \mid \rho(z)v=z^n v\}$. If $V^n\neq 0$, we say that the character z^n occurs in V.

Let $\rho\colon U_1\to \mathrm{GL}(V)$ be a representation of U_1 on a real vector space. If z^n occurs in the representation of U_1 on $V(\mathbb{C})$, so also does z^{-n}, and $V(\mathbb{C})^{-n}=\overline{V(\mathbb{C})^n}$ (complex conjugate) (otherwise the action of $\rho(z)$ on $V(\mathbb{C})$ wouldn't preserve V). On applying this with $n = 0$, we see that $V(\mathbb{C})^0$ is stable under complex conjugation, and so is defined over $\mathbb{R}$: $V(\mathbb{C})^0 = V^0(\mathbb{C})$, some $V^0 \subset V$. The natural map

$$V/V^0 \to V(\mathbb{C})/\oplus_{n\leq 0} V(\mathbb{C})^n \cong \oplus_{n>0} V(\mathbb{C})^n \tag{10}$$

is an isomorphism, and therefore makes V/V^0 into a $\mathbb{C}$-vector space.

THEOREM 1.21. *The homomorphism $u_p\colon U_1 \to G$ in 1.19 has the following properties:*

(a) *only the characters 1, z, z^{-1} occur in the representation of U_1 on $\mathrm{Lie}(G)_{\mathbb{C}}$;*

(b) $\mathrm{ad}(u(-1))$ *is a Cartan involution.*

Conversely, let (G, u) be a pair satisfying (a,b) with $u(-1) \neq 1$. There exists a pointed hermitian symmetric domain (D, p) on which $G(\mathbb{R})$ acts smoothly and $u(z)$ acts on $T_p(D)$ as multiplication by z; moreover, (D, p) is uniquely determined up to a unique isomorphism by these conditions.

PROOF. Let $L = \mathrm{Lie}(G)$. One checks that $L/L^0 \cong T_pD$. Moreover, the composite of this with $\oplus_{n>1}L(\mathbb{C})^n \cong L/L^0$ is compatible with the action of U_1. From this (a) follows, and (b) is a (nonobvious) application of (1.7).

For the converse, one lets D be the set of $G(\mathbb{R})^+$-conjugates of u (cf. §2 below). □

Thus, there is a one-to-one correspondence between isomorphism classes of pointed hermitian symmetric domains and pairs (G, u) consisting of a real adjoint Lie group and a nontrivial homomorphism $u\colon U_1 \to G(\mathbb{R})$ satisfying (a), (b).

EXAMPLE 1.22. Let $u\colon U_1 \to \mathrm{PSL}_2(\mathbb{R})$ be as in (1.20). Then $u(-1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and we saw in 1.2 that $\mathrm{ad} u(-1)$ is a Cartan involution of SL_2, hence also PSL_2.

Classification of hermitian symmetric domains in terms of Dynkin diagrams

Let H be a simple adjoint group over $\mathbb{R}$, and let u be a homomorphism $U_1 \to H$ satisfying (a) and (b) of Theorem 1.21. Because H has a compact inner form, (1.4c) implies that $H_{\mathbb{C}}$ is simple. From u we get a cocharacter $\mu = u_{\mathbb{C}}$ of $H_{\mathbb{C}}$, which satisfies the following condition:

in the action of $\mathbb{G}_m$ on $\mathrm{Lie}(H_{\mathbb{C}})$ defined by $\mathrm{ad} \circ \mu$, only the characters $z^{-1}, 1, z$ occur. (11)

PROPOSITION 1.23. *The map $(H, u) \mapsto (H_{\mathbb{C}}, u_{\mathbb{C}})$ defines bijection between the sets of isomorphism classes of pairs consisting of*

(a) *a simple adjoint group over $\mathbb{R}$ and a conjugacy class of $u\colon U_1 \to H$ satisfying (1.21a,b), and*

(b) *a simple adjoint group over $\mathbb{C}$ and a conjugacy class of cocharacters satisfying (11).*

PROOF. Let (G, μ) be as in (b), and let $g \mapsto \overline{g}$ denote complex conjugation on $G(\mathbb{C})$ relative the unique compact real form of G (see 1.3). There is real form H of G such that complex conjugation on $H(\mathbb{C}) = G(\mathbb{C})$ is $g \mapsto \mu(-1) \cdot \overline{g} \cdot \mu(-1)$, and $u =_{\mathrm{df}} \mu|U_1$ takes values in $H(\mathbb{R})$.The pair (H, u) is as in (a), the map $(G, \mu) \to (H, u)$ is inverse to $(H, u) \mapsto (H_{\mathbb{C}}, u_{\mathbb{C}})$ on isomorphism classes. □

Note that the isomorphism class of (G, μ) depends only on the conjugacy class of μ.

Let G be a simple algebraic group $\mathbb{C}$. Choose a maximal torus T in G, and a base $(\alpha_i)_{i\in I}$ for the roots of G relative to T. Recall, that the nodes of the Dynkin diagram of (G, T) are indexed by

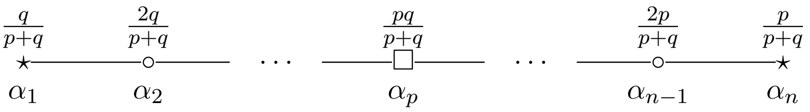
I. Recall also (Bourbaki 1981, VI 1.8) that there is a unique root $\tilde{\alpha} = \sum n_i \alpha_i$ such that, for any other root $\sum m_i \alpha_i$, $n_i \geq m_i$ for all i. An α_i (or the associated node) is said to be ***special*** if $n_i = 1$.

Let M be a conjugacy class of nontrivial cocharacters of G satisfying (11). Because all maximal tori of G are conjugate, M has a representative in $X_*(T) \subset X_*(G)$, and because the Weyl group acts simply transitively on the Weyl chambers (Serre 1987, p34; Humphreys 1972, 10.3) there is a unique representative μ for M such that $\langle \alpha_i, \mu \rangle \geq 0$. The condition (11) is that[4] $\langle \alpha, \mu \rangle \in \{1, 0, -1\}$ for all roots α. Since μ is nontrivial, not all the values $\langle \alpha, \mu \rangle$ can be zero, and so this condition implies that $\langle \alpha_i, \mu \rangle = 1$ for exactly one $i \in I$, which must in fact be special (otherwise $\langle \tilde{\alpha}, \mu \rangle > 1$). Thus, that the M satisfying (11) are in one-to-one correspondence with the special nodes of the Dynkin diagram. In conclusion:

THEOREM 1.24. *The isomorphism classes of irreducible hermitian symmetric domains are classified by the special nodes on Dynkin diagrams.*

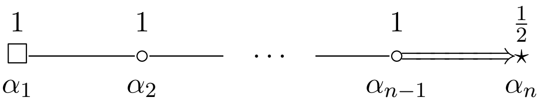
It remains to list these, with the help of the tables in Bourbaki 1981. In the following table, the special nodes are marked by squares, and the number in parentheses indicates the position of the special node. The other notations will be explained later.

$\mathbf{A_n}$: The special linear group SL_{n+1} is the subgroup of GL_{n+1} of matrices of determinant 1.

$\mathrm{A}_n(p)$: 

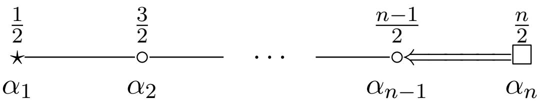
$n \geq 1$, $1 \leq p \leq n$, $p + q = n + 1$

$\mathbf{B_n}$: The special orthogonal group SO_{2n+1} is the subgroup of SL_{2n+1} of matrices A that preserve the symmetric bilinear form $\phi(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_{2n+1} y_{2n+1}$, i.e., such that $\phi(A\mathbf{x}, A\mathbf{y}) = \phi(\mathbf{x}, \mathbf{y})$.

$\mathrm{B}_n(1)$: 

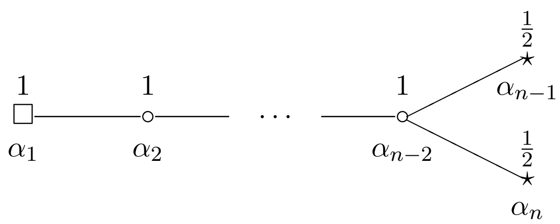
$(n \geq 2)$

$\mathbf{C_n}$: The symplectic group Sp_{2n} is the subgroup of GL_n of matrices preserving the skew-symmetric form $\psi(\mathbf{x}, \mathbf{y}) = -x_{-n} y_n - \cdots - x_{-1} y_1 + x_1 y_{-1} + \cdots + x_n y_{-n}$.

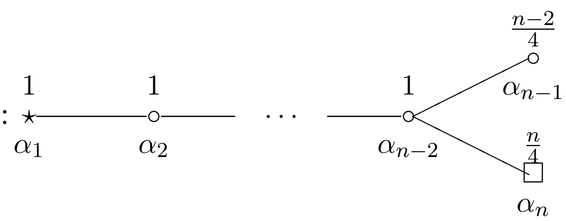
$\mathrm{C}_n(n)$: 

$(n \geq 3)$

$\mathbf{D_n}$: Same as B_n but with $2n + 1$ replaced by $2n$.

$\mathrm{D}_n(1)$: 

$(n \geq 4)$

$\mathrm{D}_n(n)$: 

$(n \geq 4)$

[4]The μ with this property are sometimes said to be ***minuscule*** (cf. Bourbaki 1981, pp226–227).

$\mathrm{D}_n(n-1)$: Same as $D_n(n)$ by with α_{n-1} and α_n interchanged (rotation about the horizontal axis).

Exceptional: There are also groups labelled E_6, E_7, E_8, F_2, G_4.

Deligne's formula. The number of isomorphism classes of hermitian symmetric domains attached to a simple adjoint group G over $\mathbb{C}$ is $i-1$ where i is the index of connectivity of G (the order of the centre of the simply connected covering group of G). This is proved by inspection. The indices are:

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$n+1$	2	2	4	3	2	1	1	1

See Bourbaki 1981, Tables pp250–275.

Irreducible hermitian symmetric domains

	EC	S	hermitian symmetric domain	dim	rank
A	I	III	$\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	pq	$\min(p,q)$
D	II	III	$\mathrm{SO}^*(2p)/\mathrm{U}(p) \quad (p>2)$	$\frac{p(p-1)}{2}$	$\left[\frac{p}{2}\right]$
C	III	I	$\mathrm{Sp}(p,\mathbb{R})/\mathrm{U}(p)$	$\frac{p(p+1)}{2}$	p
BD	IV	I	$\mathrm{SO}(p,2)^+/\left(\mathrm{SO}(p)\times\mathrm{SO}(2)\right)$	p	$\min(2,p)$
$\mathrm{E}_{6,7}$	V,VI		omitted.		

E.C. (resp. S) = the numbering of E. Cartan. (resp. Siegel).

For a square matrix A with real coefficients, $A>0$ means A is symmetric and positive definite, i.e., $x^t\cdot A\cdot x>0$ for all $x\neq 0$. For a matrix with complex coefficients, it means that A is hermitian ($A^t=\overline{A}$) and positive definite. Finally, $B>A$ means that $B-A>0$.

$\mathrm{U}(p,q)$: The group of matrices in $\mathrm{GL}(p+q,\mathbb{C})$ leaving invariant the hermitian form

$$-z_1\overline{z}_1-\cdots-z_p\overline{z}_p+z_{p+1}\overline{z}_{p+1}\cdots+z_{p+q}\overline{z}_{p+q}.$$

$\mathrm{U}(p)=U(0,p)$: The group of $g\in\mathrm{GL}_n(\mathbb{C})$ such that $\overline{g}^t\cdot g=I$.

$\mathrm{SU}(p,q)=\mathrm{U}(p,q)\cap\mathrm{SL}(p+q,\mathbb{C})$: The group of $g\in\mathrm{U}(p,q)$ with $\det g=1$.

$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$: The group of matrices $\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)$ with $a\in\mathrm{U}(p)$, $b\in\mathrm{U}(q)$, and $\det a\cdot\det b=1$.

$\mathrm{SO}^*(2p)$: The group of matrices in $\mathrm{SO}(2p,\mathbb{C})$ leaving invariant the skew-hermitian form

$$-z_1\overline{z}_{p+1}+z_{p+1}\overline{z}_1-z_2\overline{z}_{p+2}+z_{p+2}\overline{z}_2-\cdots-z_p\overline{z}_{2p}+z_{2p}\overline{z}_p.$$

$\mathrm{SO}(p,q)$: The group of matrices in $\mathrm{SL}(p+q,\mathbb{R})$ leaving invariant the quadratic form

$$-x_1^2-\cdots-x_p^2+x_{p+1}^2+\cdots+x_{p+q}^2.$$

$\mathrm{SO}(p,q)^+$: The identity component of $\mathrm{SO}(p,q)$.

$\mathrm{SO}(p)=\mathrm{SO}(0,p)=\mathrm{SO}(p,0)$.

The corresponding bounded symmetric domains and hermitian symmetric domains of compact type are:

	EC	S	bounded symmetric domain	hsd of compact type
A	I	III	$\{Z \in M_{p,q} \mid \overline{Z}^t Z < I_q\}$	$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$
D	II	III	$\{Z \in M_{p,p} \mid \overline{Z}^t Z < I_q,\, Z = -Z^t\}$	$\mathrm{SO}(2p)/\mathrm{U}(p) \quad (p > 2)$
C	III	I	$\{Z \in M_{p,p} \mid \overline{Z}^t Z < I_p,\, Z = Z^t\}$	$\mathrm{Sp}(p)/\mathrm{U}(p)$
BD	IV	I	the $z \in \mathbb{C}^p$ such that	$\mathrm{SO}(p+2)/\mathrm{SO}(p) \times \mathrm{SO}(2)$
			$\sum \lvert z_i\rvert^2 < \frac{1}{2}(1 + \lvert\sum z_i^2\rvert^2) < 1$	
E_6,$_7$	V,VI		omitted.	

NOTES. The ultimate source for hermitian symmetric domains is Helgason 1978; Wolf 1984 is also very useful. The above account is partly based on Deligne 1973 and Deligne 1979. The two tables above are based on those in Helgason 1978, p518, and Akhiezer 1990, p204.

2 Hodge structures and their classifying spaces

We describe various objects and the spaces that classify (or parametrize) them. Our goal is a description of hermitian symmetric domains as classifying spaces for certain special Hodge structures.

Review of representations

Let G be a reductive group over a field k of characteristic zero. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. The ***contragredient*** or ***dual*** $\rho^\vee$ of ρ is the representation of G on $V^\vee =_{\mathrm{df}} \mathrm{Hom}(V,k)$ defined by

$$(\rho^\vee(g)\cdot f)(v) = f(\rho(g^{-1})\cdot v), \quad g \in G,\, f \in V^\vee,\, v \in V.$$

A representation is said to be ***self-dual*** if it is isomorphic to its contragredient.

A r-***tensor*** of V is a multilinear map

$$t\colon V \times \cdots \times V \to k \quad (r\text{-copies of } V).$$

Given a representation of G on V, define gt by

$$(gt)(v_1,\ldots) = t(g^{-1}v_1,\ldots), \quad g \in G,\, (v_1,\ldots) \in V \times \cdots \times V.$$

Let $t_1,\ldots,t_m$ be tensors of V. The action of $\mathrm{GL}(V)$ on V extends to an action on V^m, and we call the stability group of $(t_1,\ldots,t_m)$ the ***subgroup of*** $\mathrm{GL}(V)$ ***fixing the tensors*** t_i.

THEOREM 2.1. *For any faithful self-dual representation $G \to \mathrm{GL}(V)$ of G, there exists a finite set T of tensors of V such that G is the subgroup of $\mathrm{GL}(V)$ fixing the $t \in T$.*

PROOF. In Deligne 1982 this is shown for T an infinite set of tensors, but clearly, a finite subset will suffice. □

For example, the symplectic group is the subgroup of $\mathrm{GL}(V)$ fixing a skew symmetric form, and the special orthogonal group is the subgroup of $\mathrm{GL}(V)$ fixing a symmetric form and the form $V \times \cdots \times V \to \bigwedge^n V \xrightarrow{\det} k$.

Let G be the subgroup of $\mathrm{GL}(V)$ fixing $t_1, \ldots, t_m$. Then

$$\begin{aligned} G(k) &= \{g \in \mathrm{GL}(V) \mid t_i(gv_1, \ldots, gv_r) = t_i(v_1, \ldots, v_r), \quad i = 1, \ldots, m\} \\ \mathrm{Lie}(G) &= \{g \in \mathrm{End}(V) \mid \textstyle\sum_{j=1}^{r} t_i(v_1, \ldots, gv_j, \ldots, v_r) = 0, \quad i = 1, \ldots, m\}. \end{aligned}$$

To prove the second equality, recall that the Lie algebra of an algebraic group G is the kernel of $G(k[\varepsilon]) \to G(k)$, $\varepsilon^2 = 1$. Thus $\mathrm{Lie}(G)$ consists of the endomorphisms $1 + g\varepsilon$ of V such that

$$t_i((1 + g\varepsilon)v_1, (1 + g\varepsilon)v_2, \ldots) = t(v_1, v_2, \ldots).$$

Expand this, and set $\varepsilon^2 = 0$.

Flag varieties

The projective space $\mathbb{P}(V)$

Let V be a finite dimensional vector space over a field k. The set $\mathbb{P}(V)$ of one-dimensional subspaces in V has a natural structure of an algebraic variety. For example, the choice of an isomorphism $V \to k^n$ determines an isomorphism $\mathbb{P}(V) \to \mathbb{P}^{n-1}$.

Grassmann varieties

2.2. Let V be a finite dimensional vector space over a field k, and let $0 < d < n = \dim V$. The set $G_d(V)$ of d-dimensional subspaces has a natural structure of an algebraic variety. In fact, the image of the map

$$W \mapsto \textstyle\bigwedge^d W \colon G_d(V) \to \mathbb{P}(\textstyle\bigwedge^d V) \tag{12}$$

has closed image (Humphreys 1975, 1.8). This map can be made explicit by fixing a basis for V. The choice of basis for W then determines a $d \times n$ matrix $A(W)$, and changing the basis for W multiplies $A(W)$ on the left by an invertible $d \times d$ matrix. Thus, the family of $d \times d$ minors of $A(W)$ is well-determined up to multiplication by a nonzero constant, and so determines a point in $\mathbb{P}(\bigwedge^d V) = \mathbb{P}^{\binom{n}{d}}$. This is the point associated with W.

2.3. Let S be a subspace of V of complementary dimension $n-d$, and let $G_d(V)_S$ be the set of $W \in G_d(V)$ such that $W \cap S = \{0\}$. Fix a $W_0 \in G_d(V)_S$, so that $V = W_0 \oplus S$. For any $W \in G_d(V)_S$, the projection $W \to W_0$ given by this decomposition is an isomorphism, and so W is the graph of a homorphism $W_0 \to S$:

$$w \mapsto s \iff (w, s) \in W.$$

Conversely, the graph of any homomorphism $W_0 \to S$ lies in $G_d(V)_S$. Thus,

$$G_d(V)_S \cong \mathrm{Hom}(W_0, S) \cong \mathrm{Hom}(V/S, S). \tag{13}$$

The decomposition $V = W_0 \oplus S$ gives a decomposition $\mathrm{End}(V) = \left(\begin{smallmatrix} \mathrm{End}(W_0) & \mathrm{Hom}(S,W_0) \\ \mathrm{Hom}(W_0,S) & \mathrm{End}(S) \end{smallmatrix}\right)$.The isomorphisms (13) show that the group $\left(\begin{smallmatrix} 1 & 0 \\ \mathrm{Hom}(W_0,S) & 1 \end{smallmatrix}\right)$ acts simply transitively on $G_d(V)_S$.

From (13) we find that the tangent space to $G_d(V)$ at W_0,[5]

$$T_{W_0}(G_d(V)) \cong \mathrm{Hom}(W_0, S) \cong \mathrm{Hom}(W_0, V/W_0). \tag{14}$$

2.4. Because $\mathrm{GL}(V)$ acts transitively on the set of bases of V, it acts transitively on $G_d(V)$. Let $P(W_0) \subset \mathrm{GL}(V)$ be the subgroup stabilizing W_0: $P(W_0) = \{g \in \mathrm{GL}(V) \mid g(W_0) = W_0\}$. Then

$$g \mapsto gW_0\colon \mathrm{GL}(V)/P(W_0) \xrightarrow{\approx} G_d(V).$$

For any complement S to W_0,

$$P(W_0) = \begin{pmatrix} \mathrm{Aut}(W_0) & \mathrm{Hom}(S,W_0) \\ 0 & \mathrm{Aut}(S) \end{pmatrix}$$

Flag varieties

2.5. Let V be a finite-dimensional vector space over k, and let $\mathbf{d} = (d_1, \ldots, d_r)$ with $0 < d_1 < \cdots < d_r < n = \dim V$. Let $G_{\mathbf{d}}(V)$ be the set of flags

$$F: \quad V \supset V^1 \supset \cdots \supset V^r \supset 0 \tag{15}$$

with V^i a subspace of V of dimension d_i. The map $F \mapsto (V^i)\colon G_{\mathbf{d}}(V) \to \prod_i G_{d_i}(V)$ realizes $G_{\mathbf{d}}(V)$ as a closed subset $\prod_i G_{d_i}(V)$, and so it is a projective variety (Humphreys 1978, 1.8). The tangent space to $G_{\mathbf{d}}(V)$ at the flag F consists of compatible families of homomorphisms $\varphi^i\colon V^i \to V/V^i$, $\varphi^i | V^{i+1} \equiv \varphi^{i+1} \mod V^{i+1}$, (Harris 1992, 16.3):

$$\begin{array}{ccccccc} \ldots & \hookleftarrow & V^i & \hookleftarrow & V^{i+1} & \hookleftarrow & \ldots \\ & & \downarrow{\scriptstyle\varphi^i} & & \downarrow{\scriptstyle\varphi^{i+1}} & & \\ \ldots & \twoheadrightarrow & V/V^i & \twoheadrightarrow & V/V^{i+1} & \twoheadrightarrow & \ldots \end{array} \tag{16}$$

Again, $\mathrm{GL}(V)$ acts transitively on $G_{\mathbf{d}}(V)$, and so, for any flag F_0,

$$g \mapsto gF_0\colon \mathrm{GL}(V)/P(F_0) \xrightarrow{\approx} G_{\mathbf{d}}(V)$$

where $P(F_0)$ is the subgroup of $\mathrm{GL}(V)$ stabilizing F_0.

Hodge structures

Hodge decompositions

2.6. For a real vector space V, complex conjugation on $V(\mathbb{C}) =_{\mathrm{df}} \mathbb{C} \otimes_{\mathbb{R}} V$ is defined by

$$\overline{z \otimes v} = \bar{z} \otimes v.$$

[5] The composite $G_d(V)_S \cong \mathrm{Hom}(V/S, S)$ in (13) depends on the choice of W_0. A more precise statement is that $G_d(V)_S$ is an affine space (principal homogeneous space) for $\mathrm{Hom}(V/S, S)$. On the other hand, the isomorphism $T_{W_0}(G_d(V)) \cong \mathrm{Hom}(W_0, V/W_0)$ is independent of the choice of the complement S to W_0.

An $\mathbb{R}$-basis $(e_i)_{1\leq i\leq m}$ for V is also a $\mathbb{C}$-basis for $V(\mathbb{C})$, and $\overline{\sum a_i e_i} = \sum \overline{a_i} e_i$. A ***Hodge decomposition of weight*** n of a real vector space V is a decomposition $V(\mathbb{C}) = \oplus_{p+q=n} V^{p,q}$ such that $V^{p,q}$ is the complex conjugate of $V^{p,q}$. An ***integral*** (resp. ***rational***, resp. ***real***) ***Hodge structure*** is a free $\mathbb{Z}$-module of finite rank V (resp. finite-dimensional $\mathbb{Q}$-vector space; resp. finite-dimensional $\mathbb{R}$-vector space) together with a Hodge decomposition of $V(\mathbb{R})$. The set of pairs (p,q) for which $V^{p,q} \neq 0$ is called the ***type*** of the Hodge structure.

2.7. Let J be a complex structure on a real vector space V (i.e., a $\mathbb{R}$-linear map such that $J^2 = -1$), and define $V^{1,0}$ and $V^{0,1}$ to be the $-i$ and $+i$ eigenspaces of J acting on $V(\mathbb{C})$. Then $V(\mathbb{C}) = V^{1,0} \oplus V^{0,1}$ is a Hodge structure of type $(1,0)$, $(0,1)$, and every Hodge structure of this type arises from a (unique) complex structure.

Morphisms of Hodge structures

2.8. Let $(V,(V^{p,q}))$ and $(W,(W^{p,q}))$ be Hodge structures of weights m and n respectively. A ***morphism*** $(V,(V^{p,q})) \to (W,(W^{p,q}))$ is a linear map $\alpha\colon W \to V$ such that $\alpha_{\mathbb{C}}(W^{p,q}) \subset V^{p,q}$ for all p,q. Thus, $\alpha = 0$ if $m \neq n$. The vector space (or free $\mathbb{Z}$-module) $\mathrm{Hom}(V,W)$ acquires a Hodge structure with

$$\mathrm{Hom}(V,W)^{r,s} = \{\alpha \in \mathrm{Hom}(V(\mathbb{C}),W(\mathbb{C})) \mid \alpha(V^{p,q}) \subset W^{p+r,q+s}\}.$$

Then

$$\mathrm{Hom}((V,(V^{p,q})),(W,(W^{p,q})) = \mathrm{Hom}(V,W) \cap \mathrm{Hom}(V,W)^{0,0}.$$

For example, a morphism of real Hodge structures of type $(1,0)$, $(0,1)$ is a homomorphism of real vector space commuting with J, i.e., that is $\mathbb{C}$-linear (2.7).

The Hodge filtration

2.9. The ***Hodge filtration*** associated with a Hodge structure of weight n is

$$F^\bullet\colon \quad \cdots \supset F^p \supset F^{p+1} \supset \cdots, \quad F^p = \oplus_{r\geq p} V^{r,s}.$$

Note that

$$\overline{F^q} = \oplus_{s\geq q} \overline{V^{s,r}} = \oplus_{s\geq q} V^{r,s} = \oplus_{r\leq n-q} V^{r,s} \overset{\text{if } q=n+1-p}{=} \oplus_{r\leq p-1} V^{r,s},$$

and so $V(\mathbb{C})$ is the direct sum of F^p and $\overline{F^q}$ whenever $p+q = n+1$. Conversely, if $F^\bullet$ is a finite descending filtration of $V(\mathbb{C})$ such that

$$V(\mathbb{C}) = F^p \oplus \overline{F^q} \text{ whenever } p+q = n+1, \tag{17}$$

then $F^\bullet$ defines a Hodge structure of weight n by the rule $V^{p,q} = F^p \cap \overline{F^q}$.

2.10. For example, for a Hodge structure of type $(1,0)$, $(0,1)$, the Hodge filtration is

$$F^0 = V(\mathbb{C}) \supset F^1 = V^{1,0} \supset F^2 = 0$$

(and the isomorphism $V \to V/F^0$ defines the complex structure on V noted in (2.7)).

2.11. Let $(V_o^{p,q})_{p+q=n}$ be a Hodge decomposition of V. Since the Hodge filtration determines the Hodge decomposition $(V^{p,q})_{p+q=n} \mapsto F^\bullet$ determines a bijection from the set of Hodge decompositions $(V^{p,q})_{p+q=n}$ with $\dim V^{p,q} = \dim V_o^{p,q}$ onto a subset of a flag manifold.

The Weil operator

2.12. Define $C\colon V(\mathbb{C}) \to V(\mathbb{C})$ to act on $V^{p,q}$ as multiplication by i^{q-p}. Then C commutes with complex conjugation on $V(\mathbb{C})$: if $x \in V^{p,q}$, then $\overline{x} \in V^{q,p}$, and so $C\overline{x} = i^{p-q}\overline{x} = \overline{i^{q-p}x}$. Thus, C is an $\mathbb{R}$-linear map $V \to V$. Note that $(i^{q-p})^2 = i^{2q-2p+4p} = (-1)^n$, and so C^2 acts as 1 or -1 according as n is even or odd.

2.13. For example, if V is of type $(1,0)$, $(0,1)$, then C coincides with the J of (2.7). The functor $(V,(V^{1,0},V^{0,1})) \mapsto (V,C)$ is an equivalence from the category of real Hodge structures of type $(1,0)$, $(0,1)$, to the category of complex vector spaces.

Hodge structures of weight 0.

2.14. Let V be a Hodge structure of weight 0. Then $V^{0,0}$ is invariant under complex conjugation, and so $V^{0,0} = V^{00} \otimes \mathbb{C}$, where $V^{00} = V^{0,0} \cap V$. Note that

$$V^{00} = \operatorname{Ker}(V \to V(\mathbb{C})/F^0). \tag{18}$$

Hodge tensors

2.15. Let $(V,(V^{p,q}))$ be a real Hodge structure of weight n. Let $t\colon V \times \cdots \times V \to \mathbb{R}$ be a multilinear map (r-tensor for V). We say that t is a ***Hodge tensor*** if

$$\textstyle\sum p_i \neq \sum q_i \Rightarrow t_{\mathbb{C}}(v_1^{p_1,q_1}, v_2^{p_2,q_2}, \ldots) = 0. \tag{19}$$

Note that, for a Hodge tensor

$$t(Cv_1, Cv_2, \ldots) = t(v_1, v_2, \ldots)$$

because, if $t_{\mathbb{C}}(v_1^{p_1,q_1}, v_2^{p_2,q_2}, \ldots) \neq 0$, then C multiplies it by $i^{\sum q_i - \sum p_i} = 1$.

ASIDE 2.16. The ***tensor product*** of Hodge structures $(V,(V^{p,q}))$ and $(W,(W^{p,q}))$ of weights m and n is the Hodge structure $(V \otimes W, ((V \otimes W)^{p,q}))$ of weight $m+n$ with

$$(V \otimes W)^{p,q} = \oplus_{r+r'=p, s+s'=q} V^{r,s} \otimes W^{r',s'}.$$

Let $\mathbb{R}(m)$ be the unique Hodge structure of weight $-2m$ on $V = \mathbb{R}$; thus $\mathbb{R}(m)^{-m,-m} = \mathbb{C}$. To give a multilinear map $t\colon V \times \cdots \times V \to \mathbb{R}$ amounts to giving a homomorphism $t^{\otimes}\colon V \otimes \cdots \otimes V \to \mathbb{R}$, and t is a Hodge tensor if and only if $t^{\otimes}$ is a morphism of Hodge structures $V \otimes \cdots \otimes V \to \mathbb{R}(-nr)$.

Polarizations

2.17. Let ψ be a bilinear form on a real vector space V. A Hodge structure $(V^{p,q})_{p,q}$ of weight n on V is ***positive for*** ψ (and ψ is a ***polarization of*** $(V,(V^{p,q}))$) if ψ is a Hodge tensor and $\psi_C(u,v) =_{\text{df}} \psi(u,Cv)$ is symmetric and positive definite. Note that, because ψ is a Hodge tensor

$$\psi(Cu, Cv) = \psi(u,v).$$

Moreover, ψ is symmetric or skew symmetric according as n is even or odd, because $\psi(v,u) = \psi(Cv,Cu) = \psi(u,C^2v) = (-1)^n\psi(u,v)$ (2.12).

2.18. For example, a polarization of a Hodge structure V of type $(1,0)$, $(0,1)$, is a skew symmetric form $\psi\colon V\times V\to\mathbb{R}$ such that $\psi(Ju,Jv)=\psi(u,v)$ and $\psi(u,Jv)$ is symmetric and positive definite.

Hodge structures as representations of $\mathbb{S}$

2.19. Let $\mathbb{S}$ be $\mathbb{C}^{\times}$ regarded as a torus over $\mathbb{R}$, i.e., $\mathbb{S}=\operatorname{Spec}\mathbb{R}[X,Y,T]/((X^2+Y^2)T-1)$ with the obvious group structure. There are homomorphisms

$$\begin{array}{ccccccccccc} \mathbb{G}_m & \xrightarrow{w} & \mathbb{S} & \xrightarrow{t} & \mathbb{G}_m & \qquad & U_1 & \longrightarrow & \mathbb{S} & \longrightarrow & U_1 \\ \mathbb{R}^{\times} & \xrightarrow{r\mapsto r} & \mathbb{C}^{\times} & \xrightarrow{z\mapsto(z\overline{z})^{-1}} & \mathbb{R}^{\times} & \qquad & U_1 & \xrightarrow{z\mapsto z} & \mathbb{C}^{\times} & \xrightarrow{z\mapsto z/\overline{z}} & U_1. \end{array} \tag{20}$$

Note that $\mathbb{S}(\mathbb{C})\approx\mathbb{C}^{\times}\times\mathbb{C}^{\times}$ with complex conjugation acting by $\overline{(z_1,z_2)}=(\overline{z_2},\overline{z_1})$. We fix the isomorphism $\mathbb{S}_{\mathbb{C}}\cong\mathbb{G}_m\times\mathbb{G}_m$ so that $\mathbb{S}(\mathbb{R})\to\mathbb{S}(\mathbb{C})$ is $z\mapsto(z,\overline{z})$. Then,

$$\begin{array}{ccccccccccc} \mathbb{G}_m & \xrightarrow{w_{\mathbb{C}}} & \mathbb{S}_{\mathbb{C}} & \xrightarrow{t_{\mathbb{C}}} & \mathbb{G}_m & \qquad & U_{1\mathbb{C}} & \longrightarrow & \mathbb{S}_{\mathbb{C}} & \longrightarrow & U_{1\mathbb{C}} \\ \mathbb{C}^{\times} & \xrightarrow[(z,z)]{z\mapsto} & \mathbb{C}^{\times}\times\mathbb{C}^{\times} & \xrightarrow[(z_1z_2)^{-1}]{(z_1,z_2)\mapsto} & \mathbb{C}^{\times} & \qquad & \mathbb{C}^{\times} & \xrightarrow[(z,z^{-1})]{z\mapsto} & \mathbb{C}^{\times}\times\mathbb{C}^{\times} & \xrightarrow[z_1/z_2]{(z_1,z_2)\mapsto} & \mathbb{C}^{\times} \end{array} \tag{21}$$

Also, there is a homomorphism

$$\mu\colon\mathbb{G}_m\to S_{\mathbb{C}},\quad \mathbb{C}^{\times}\xrightarrow{z\mapsto(z,1)}\mathbb{C}^{\times}\times\mathbb{C}^{\times}. \tag{22}$$

2.20. Let V be an $\mathbb{R}$-vector space. The characters of $\mathbb{S}_{\mathbb{C}}$ are the homomorphisms $(z_1,z_2)\mapsto z_1^pz_2^q$, $(p,q)\in\mathbb{Z}$. Therefore, to give a homomorphism $h\colon\mathbb{S}_{\mathbb{C}}\to\mathrm{GL}(V(\mathbb{C}))$ is the same as to give a decomposition

$$V=\oplus V^{p,q},\quad V^{p,q}=\{v\in V\mid h(z)v=z_1^qz_2^pv\}.$$

Note the switch in the p and q! The homomorphism h is defined over $\mathbb{R}$ if and only if $\overline{V^{p,q}}=V^{q,p}$, all p,q. Therefore, to give a Hodge structure of weight n on V is the same as to give a homomorphism $h\colon\mathbb{S}\to\mathrm{GL}(V)$ such that $h\circ w(r)=r^n$. Under our convention, $h(z)$ acts on $V^{p,q}$ as $\overline{z}^pz^q$. A tensor of V is a Hodge tensor if and only if it is fixed by $h(\mathbb{C}^{\times})$.

2.21. For a Hodge structure of type $(1,0)$, $(0,1)$, the isomorphism $V\to V(\mathbb{C})/F^1$ makes V into a $\mathbb{C}$-vector space, and $h(z)v=zv$ for this structure (which is the reason for our convention).

Variations of Hodge structures

2.22. Fix a real vector space V, and let S be a connected complex manifold. A family of Hodge structures $(V_s^{p,q})_{s\in S}$ on V parametrized by S is said to be ***continuous*** if, for each p,q, the $V_s^{p,q}$ vary continuously with s, i.e., $d(p,q)=\dim V^{p,q}$ is constant, and $s\mapsto V_s^{p,q}\colon S\to G_{d(p,q)}(V)$ is continuous. Let $\mathbf{d}=(\ldots,d(p),\ldots)$ with $d(p)=\sum_{r\geq p}d(r,s)$. A continuous family of Hodge structures $(V_s^{p,q})_s$ is ***holomorphic*** if the Hodge filtrations $F_s^\bullet$ vary holomorphically, i.e.,

$$s \mapsto F_s^\bullet \colon S \xrightarrow{\varphi} G_{\mathbf{d}}(V)$$

is holomorphic. The differential of φ at s is a $\mathbb{C}$-linear map (see (16))

$$d\varphi_s \colon T_s S \to T_{F_s^\bullet}(G_{\mathbf{d}}(V)) \subset \oplus_p \operatorname{Hom}(F_s^p, V/F_s^p).$$

If the image of $d\varphi_s$ is contained in

$$\oplus_p \operatorname{Hom}(F_s^p, F_s^{p-1}/F_s^p),$$

for all s, then the holomorphic family is called a ***variation of Hodge structures on*** S.

Let V be a real vector space, and let $T = (t_i)_{0\le i\le r}$ be a family of tensors with t_0 a nondegenerate bilinear form on V. Let $d\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{N}$ be such that

$$\begin{aligned} &d(p,q) = 0 \text{ for almost all } p,q;\\ &d(q,p) = d(p,q);\\ &d(p,q) = 0 \text{ unless } p+q=n. \end{aligned}$$

Let $S(d,T)$ be the set of all Hodge structures $(V^{p,q})_{p,q}$ on V such that

- $\dim V^{p,q} = d(p,q)$ for all p,q;
- each $t\in T$ is a Hodge tensor for $(V^{p,q})_{p,q}$;
- t_0 is a polarization for $(V^{p,q})_{p,q}$.

Then $S(d,T)$ acquires a topology as a subspace of $\prod_{d(p,q)\neq 0} G_{d(p,q)}(V)$. Let $S(d,T)^+$ be a connected component of $S(d,T)$.

THEOREM 2.23. (a) *If nonempty, $S(d,T)^+$ has a unique structure of a complex manifold for which $(V,(V_s^{p,q})_{p,q})$ is a holomorphic family of Hodge structures.*

(b) *With this complex structure, $S(d,T)^+$ is a hermitian symmetric domain if and only if $(V,(V_s^{p,q}))$ is a variation of Hodge structures.*

(c) *Every irreducible hermitian symmetric domain is of the form $S(d,T)^+$ for a suitable V, d, and T.*

PROOF. (a) Let $S^+ = S(d,T)^+$. Because the Hodge filtration determines the Hodge decomposition, the map $x \mapsto F_s^\bullet \colon S^+ \xrightarrow{\varphi} G_{\mathbf{d}}(V)$ is injective. Let G be the subgroup of $\mathrm{GL}(V)$ fixing the $t\in T$, and let $H_o \in S^+$. It can be shown that

$$S^+ = G(\mathbb{R})^+ \cdot H_o.$$

The subgroup $G(\mathbb{R})^+_o$ of $G(\mathbb{R})^+$ fixing H_o is closed, and so

$$S^+ = (G(\mathbb{R})^+/G(\mathbb{R})^+_o)\cdot H_o \cong G(\mathbb{R})^+/G(\mathbb{R})^+_o$$

is a real analytic manifold. Let $L = \mathrm{Lie}(V)$. It acts on V, and we have a diagram

$$\begin{array}{ccc} T_{H_o}S^{+} \cong \; L/L^{00} & \hookrightarrow & \mathrm{End}(V)/\mathrm{End}(V)^{00} \\ {\scriptstyle (18)}\downarrow\cong & & {\scriptstyle (18)}\downarrow\cong \\ L(\mathbb{C})/F^{0} & \hookrightarrow & \mathrm{End}(V(\mathbb{C}))/F^{0} \;\; \cong T_{H_o}G_{\mathbf{d}}(V) \end{array} \tag{23}$$

Thus, $d\varphi_{H_o}$ maps $T_{H_o}S^{+}$ onto a complex subspace of $T_{H_o}G_{\mathbf{d}}(V)$, and it follows that φ maps S^{+} onto a complex submanifold of $G_{\mathbf{d}}(V)$.

(b) Let h_o be the homomorphism $\mathbb{C}^{\times} \to \mathrm{GL}(V)$ defined by H_o (2.20). Because the t_i are Hodge tensors, $h_o(\mathbb{S}) \subset G$. We have $gh_og^{-1} = h_{go}$. Therefore, $h_o(r) \in Z(G)$. Define $u_o\colon U_1 \to G^{\mathrm{ad}}$ by $u_o(z) = h_o(\sqrt{z})$. Let $C = h_o(i) = u_o(-1)$. Then the faithful representation $G \to \mathrm{GL}(V)$ carries a C-polarization, and so $\mathrm{ad}C$ is a Cartan involution on G (hence on G^{ad}) (1.7). One checks from (23) that u_o satisfies (a) of Theorem 1.21 if and only if $(V_s^{p,q})$ is a variation of Hodge structures. Since

$$\{G^{\mathrm{ad}}(\mathbb{R})^{+}\text{-conjugacy class of } u_o\} \overset{\cong}{\leftarrow} \{G(\mathbb{R})^{+}\text{-conjugacy class of } h_o\} \overset{\cong}{\rightarrow} S(d,T),$$

this proves (b).

(c) Given an irreducible hermitian symmetric domain D, choose a faithful self-dual representation $G \to \mathrm{GL}(V)$ of the algebraic group G associated with D (so G is such that $G(\mathbb{R})^{+} = \mathrm{Aut}(D)^{+}$). Because V is self-dual, there is a nondegenerate bilinear form t_0 on V fixed by G. Apply Theorem 2.1 to find tensors $t_1, \ldots, t_n$ such that G is the subgroup of $\mathrm{GL}(V)$ fixing $t_0, \ldots, t_n$. Let h_p be the composite $\mathbb{S} \overset{z \mapsto z/\bar{z}}{\to} U_1 \overset{u_p}{\to} \mathrm{GL}(V)$ with u_p as in (1.19). Then, h_p defines a Hodge structure on V for which the t_i are Hodge tensors and t_0 is a polarization. One can check that D is naturally identified with the component of $S(d, \{t_0, t_1, \ldots, t_n\})^{+}$ containing this Hodge structure. □

REMARK 2.24. Given a pair $(V, (V^{p,q})_{p,q}, T)$, define L to be the sub-Lie-algebra of $\mathrm{End}(V)$ fixing the $t \in T$, i.e., such that

$$\textstyle\sum_i t(v_1, \ldots, gv_i, \ldots, v_r) = 0.$$

Then L has a Hodge structure of weight 0. We say that $(V, (V^{p,q})_{p,q}, T)$ is ***special*** if L is of type $(-1,1), (0,0), (1,-1)$. The family $S(d,T)^{+}$ containing $(V, (V^{p,q})_{p,q}, T)$ is a variation of Hodge structures if and only if (H,T) is special.

ASIDE 2.25. To be added: explain the geometric significance of all this (Hodge, Griffiths, Deligne).

3 Locally symmetric varieties

Quotients of symmetric hermitian domains by discrete groups

Recall that a group Γ acts ***freely*** on a set D if $\gamma x = x$ implies $\gamma = 1$.

PROPOSITION 3.1. *Let D be a symmetric hermitian domain, and let Γ be a discrete subgroup of $A =_{df} \mathrm{Aut}(D)^{+}$. If Γ acts freely on D, then $\Gamma\backslash D$ becomes a manifold for the quotient topology, and it has a unique complex structure for which a function f on an open subset of U of $\Gamma\backslash D$ is holomorphic if and only if $f \circ \pi$ is holomorphic on $\pi^{-1}U$. Here π is the quotient map $D \to \Gamma\backslash D$.*

PROOF. The proof is essentially the same as in the case $D = \mathcal{H}_1$. Both A and D are locally compact and Hausdorff, there is a countable basis for the topology on A, and A acts continuously and transitively on D. Therefore, for any $x \in D$, the map $a \mapsto ax \colon A \to D$ defines a homeomorphism $A/\operatorname{Stab}(x) \to D$ (Milne MF, 1.2).

Because Γ is discrete in A, it acts properly discontinuously on D (ibid. 2.4). In fact, for points $x, y \in D$ not in the same Γ-orbit, there exist neighbourhoods U of x and V of y such that $\gamma U \cap V = \emptyset$ for all $\gamma \in \Gamma$ (ibid. 2.5c). Then πU and πV are disjoint neighbourhoods of πx and πy, and so $\Gamma\backslash D$ is Hausdorff.

Let $z \in \Gamma\backslash D$, and let $x \in \pi^{-1}(z)$. Because Γ acts freely on D, there is a neighbourhood U of x such that γU is disjoint from U for all $\gamma \neq 1$ (ibid. 2.5b). The restriction of π to U is a homeomorphism $U \to \pi U$. Evidently, we can choose U to be a coordinate neighbourhood (U, u) of x, and then $(\pi U, u \circ (\pi|U)^{-1})$ is a coordinate neighbourhood of z. One checks easily that the coordinate neighbourhoods obtained in this way are compatible. □

We write $D(\Gamma)$ for $\Gamma\backslash D$ with its complex structure.

REMARK 3.2. The map $\pi \colon D \to D(\Gamma)$ realizes $D(\Gamma)$ as the quotient of D by Γ in the category of complex manifolds, i.e., π is holomorphic, and a map $\varphi \colon D(\Gamma) \to M$ from $D(\Gamma)$ to a complex manifold M is holomorphic if $\varphi \circ \pi$ is holomorphic. [Proof: Let φ be a map such that $\varphi \circ \pi$ is holomorphic, and let f be a holomorphic function on an open subset U of M. Then $f \circ \varphi$ is holomorphic because $f \circ \varphi \circ \pi$ is holomorphic. Thus, φ is a morphism of ringed spaces, which is what we mean by a holomorphic map of complex manifolds.]

Subgroups of finite covolume

We shall be interested in quotients of D only by "big" discrete subgroups Γ of $\mathrm{Aut}(D)^{+}$. This condition is conveniently expressed by saying that $D(\Gamma)$ has finite volume. By definition, D has a riemannian metric g and hence a volume element Ω: in local coordinates

$$\Omega = \sqrt{\det(g_{ij}(x))}dx^1 \wedge \ldots \wedge dx^n.$$

Since g is invariant under Γ, so also is Ω, and therefore passes to the quotient $\Gamma\backslash D$. The condition is that

$$\mathrm{vol(D(\Gamma))} \stackrel{\text{df}}{=} \int_{\Gamma\backslash D} \Omega < \infty.$$

For example, let $D = \mathcal{H}_1$ and let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Then

$$F = \{z \in \mathcal{H}_1 \mid |z| > 1, \quad -\frac{1}{2} \leq \Re \leq \frac{1}{2}\}$$

is a fundamental domain for Γ and

$$\mathrm{vol}(\Gamma \backslash \mathrm{D}) = \int_{\Gamma \backslash D} \Omega = \iint_F \frac{dxdy}{y^2} \leq \int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2} \frac{dxdy}{y^2} = \int_{\sqrt{3}/2}^{\infty} \frac{dy}{y^2} < \infty.$$

On the other hand, the quotient of $\mathcal{H}_1$ by the group $\Gamma = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | \, n \in \mathbb{Z}\}$ of translations $z \mapsto z + n$, $n \in \mathbb{Z}$, has infinite volume, as does the quotient of $\mathcal{H}_1$ by $\Gamma = 1$.

ASIDE 3.3. A real Lie group G has a left invariant volume element, which is unique up to a positive constant (cf. Boothby 1975, VI 3.5). A ***lattice*** in G is a discrete subgroup Γ such that $\Gamma \backslash G$ has finite volume.

Let K be a compact subgroup of G. There is a G-invariant Borel measure ν on G/K that is finite on compact subgroups, and ν is unique up to a positive constant. Now suppose G/K has the structure of a hermitian symmetric domain X. Then the riemannian volume form on X has the properties of ν. An application of Fubini's theorem shows that, for a discrete subgroup Γ of G acting freely on X, $\Gamma \backslash G$ has finite volume if and only if $\Gamma \backslash X$ has finite volume. Thus, the discrete subgroups Γ of G acting freely on X for which $\Gamma \backslash X$ has finite volume are precisely the lattices in G that act freely on X (cf. Witte 2001, §1).

Arithmetic subgroups

Two subgroups A and B of a group are ***commensurable*** if $A \cap B$ has finite index in both A and B. For example, the infinite cyclic subgroups $a\mathbb{Z}$ and $b\mathbb{Z}$ of $\mathbb{R}$ are commensurable if and only if $a/b \in \mathbb{Q}^{\times}$. Commensurability is an equivalence relation (obvious, except for transitivity).

Let G be an algebraic group over $\mathbb{Q}$. Then G can be realized as a closed subgroup of GL_n for some n. Then any subgroup of $G(\mathbb{Q})$ commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ is is said to be ***arithmetic***. One can show that this definition is independent of the choice $G \hookrightarrow \mathrm{GL}_n$.

Let A be a connected real Lie group. A subgroup Γ of A is ***arithmetic*** if there exists an algebraic group G over $\mathbb{Q}$ and an arithmetic subgroup Γ_0 of $G(\mathbb{Q})$ such that $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto Γ under a surjective homomorphism $G(\mathbb{R})^+ \to A$ with compact kernel.

PROPOSITION 3.4. *Let $\rho\colon G \to G'$ be a surjective homomorphism of algebraic groups over $\mathbb{Q}$. If $\Gamma \subset G(\mathbb{Q})$ is arithmetic, then so also is $\rho(\Gamma) \subset G'(\mathbb{Q})$.*

PROOF. Borel 1966, 1.2, or Borel 1969, 8.9.[6] □

PROPOSITION 3.5. *Let A be a connected real semisimple Lie group. Then an arithmetic subgroup Γ of A is discrete in A and of finite covolume. Moreover, if $A = \mathrm{Aut}(D)^+$ for D a hermitian symmetric domain, then every torsion-free arithmetic subgroup of A acts freely on D.*

[6]Borel, A., 1969, Introduction aux groupes arithmétiques, Hermann.

PROOF. Omit (cf. Witte 2001, 6.10). □

REMARK 3.6. An endomorphism α of a $\mathbb{Q}$-vector space V is said to be ***neat*** if its eigenvalues in $\mathbb{Q}^{\mathrm{al}}$ generate a torsion-free subgroup of $\mathbb{Q}^{\mathrm{al}\times}$. An element $g \in G(\mathbb{Q})$ is ***neat*** if $\rho(g)$ is neat for one faithful (hence every) representation. A subgroup of $G(\mathbb{Q})$ is ***neat*** if all its elements are. Every arithmetic subgroup of $G(\mathbb{Q})$ has a neat subgroup of finite index (Borel 1969, 17.6). Such a subgroup is obviously torsion-free.

REMARK 3.7. There are many nonarithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$. Recall (Riemann mapping theorem) than every simply connected riemann surface is isomorphic to exactly one of (a) the riemann sphere, (b) $\mathbb{C}$, or (c) $\mathcal{H}_1$. The first two are the universal covering spaces of the complete nonsingular curves of genus 0 and genus 1 respectively. It follows that every complete nonsingular curve of genus $g \geq 2$ is the quotient of $\mathcal{H}_1$ by a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^+$ acting freely on $\mathcal{H}_1$. Since there are continuous families of such curves, this shows that there are uncountably many lattices in $\mathrm{PGL}_2(\mathbb{R})^+$ (therefore in $\mathrm{SL}_2(\mathbb{R})$), but there only countably many arithmetic subgroups.

REMARK 3.8. The following (Fields medal) theorem of Margulis et al. show that SL_2 is exceptional in this regard: if G is not isogenous to $\mathrm{SO}(1,n) \times \mathrm{compact}$ or $\mathrm{SU}(1,n) \times \mathrm{compact}$ and Γ is irreducible, then Γ is arithmetic — see Margulis 1991 (for the proof) or Witte 2001, 6.21 (for a discussion). A lattice Γ in a connected semisimple group G without compact factors is ***reducible*** if there exist normal connected subgroups H and H' in G such that $HH' = G$, $H \cap H'$ is discrete, and $\Gamma/(\Gamma \cap H) \cdot (\Gamma \cap H')$ is finite; otherwise, it is ***irreducible***. As $\mathrm{SL}_2(\mathbb{R})$ is isogenous to $\mathrm{SO}(1,2)$, the theorem doesn't apply to it.

Algebraic varieties versus complex manifolds

For algebraic varieties, I use the conventions of Milne AG. In particular, an algebraic variety is a ringed space whose points are closed. A morphism of algebraic varieties is called a ***regular map***.

For a nonsingular variety V over $\mathbb{C}$, $V(\mathbb{C})$ has a natural structure as a complex manifold. More precisely:

PROPOSITION 3.9. *There is a unique functor $(V, \mathcal{O}_V) \mapsto (V^{an}, \mathcal{O}_{V^{an}})$ from nonsingular varieties over $\mathbb{C}$ to complex manifolds with the following properties:*

(a) *as sets, $V = V^{an}$, and every Zariski open subset is open for the complex topology;*
(b) *if $V = \mathbb{A}^n$, then $V^{an} = \mathbb{C}^n$ with its natural structure as a complex manifold;*
(c) *if $\varphi\colon V \to U$ is étale, then $\varphi^{an}\colon V^{an} \to U^{an}$ is a local isomorphism.*

PROOF. Recall that regular map $\varphi\colon V \to U$ is étale if the map $d\varphi_P\colon T_PV \to T_PU$ is an isomorphism for all $P \in V$. Note that conditions (b,c) determine the complex-manifold structure on any variety V that admits an étale map to an open subvariety of $\mathbb{A}^n$. Since every nonsingular variety admits an open covering by such V (Milne AG, 4.31), this shows that there exists at most one functor satisfying (a,b,c), and suggests how to define it. □

Obviously, a regular map $\varphi\colon V \to W$ is determined by $\varphi^{\mathrm{an}}\colon V^{\mathrm{an}} \to W^{\mathrm{an}}$, but not every holomorphic map $V^{\mathrm{an}} \to W^{\mathrm{an}}$ is regular. For example, $z \mapsto e^z\colon \mathbb{C} \to \mathbb{C}$ is not regular.

A complex manifold need not arise from a nonsingular algebraic variety, and when it does it may arise from more than one. In other words, two nonsingular varieties V and W may be isomorphic as complex manifolds without being isomorphic as varieties.

There two basic reasons why a complex manifold may not arise from an algebraic variety:

(a) it may have no “nice” compactification, or

(b) it may have too few functions.

One positive result: the functor

$$\{\text{projective nonsingular curves over } \mathbb{C}\} \to \{\text{compact riemann surfaces}\}$$

is an equivalence of categories. Since the proper Zariski closed subsets of algebraic curves are the finite subsets, we see that the functor

$$\{\text{affine nonsingular curves over } \mathbb{C}\} \to \{\text{compact riemann surfaces} \smallsetminus \text{finite sets}\}$$

is also an equivalence. In particular, we see that, if a riemann surface U arises from an algebraic curve, then any bounded holomorphic function on U is constant (because it will extend to a holomorphic function on the compact riemann surface containing U). We can conclude that $\mathcal{H}_1$ doesn't arise from an algebraic curve because, for example, the function $z \mapsto \frac{z-i}{z+i}$ is bounded, holomorphic, and nonconstant. In fact, this function is an isomorphism onto $D_1 = \{z \mid |z| < 1\}$. Note that D_1 has a canonical compactification, namely, the closed unit disk, but this makes no sense algebraically (the closed disk has complex dimension 1, while it boundary has complex dimension $1/2$; it is not possible to complete D_1 to a compact complex manifold by adding a finite number of points).

Recall that, for any full lattice Λ in $\mathbb{C}$, the Weierstrass $\wp$ function and its derivative embed $\mathbb{C}/\Lambda$ into $\mathbb{P}^2(\mathbb{C})$ (as an elliptic curve). For a full lattice Λ in $\mathbb{C}^2$, the field of meromorphic functions on $\mathbb{C}^2/\Lambda$ will usually have transcendence degree < 2, and so $\mathbb{C}^2/\Lambda$ can't be an algebraic variety. (For an algebraic variety V of dimension n over $\mathbb{C}$, the field of rational functions on V has transcendence degree n, and, if V is nonsingular, the rational functions are meromorphic on the complex manifold V^{an}.)

A complex manifold (resp. algebraic variety) is ***projective*** if it is isomorphic to a closed submanifold (resp. closed subvariety) of projective space. The first truly satisfying theorem in the subject is the following:

THEOREM 3.10 (CHOW AJM 1949). *Every projective complex manifold has a unique structure of a projective algebraic variety, and every holomorphic map of projective complex manifolds is regular for these structures.*

PROOF. See Shafarevich 1994, VIII, 3.1. □

In other words, the functor $V \mapsto V^{\mathrm{an}}$ is an equivalence of the category of projective algebraic varieties with the category of projective complex manifolds.

The discussion so far should suggest that the question of algebraicity for a complex manifold is closely related to the question of a good compactification.

The theorem of Baily and Borel

THEOREM 3.11 (BAILY-BOREL). *Let $D(\Gamma) = \Gamma\backslash D$ be the quotient of a hermitian symmetric domain by a torsion-free arithmetic subgroup Γ. Then $D(\Gamma)$ has a canonical realization as a Zariski-open subset of a projective algebraic variety $D(\Gamma)^*$. In particular, it has a canonical structure as an algebraic variety.*

PROOF. Recall the proof for $D = \mathcal{H}_1$. Set $\mathcal{H}_1^* = \mathcal{H}_1 \cup \mathbb{P}^1(\mathbb{Q})$ (rational points on the real axis plus the point $i\infty$). Then Γ acts on $\mathcal{H}_1^*$ and $\Gamma\backslash\mathcal{H}_1^*$ is a compact riemann surface. One then shows that there enough modular forms on $\mathcal{H}_1$ to embed $\Gamma\backslash\mathcal{H}_1^*$ in some projective space, and $\Gamma\backslash\mathcal{H}_1$ is a Zariski-open subset of $\Gamma\backslash\mathcal{H}_1^*$ because it omits only finitely many points. In outline, the proof in the general case is similar, but is **much** harder (Baily & Borel 1966). □

REMARK 3.12. (a) The variety $D(\Gamma)^*$ is very singular. The boundary $D(\Gamma)^* \smallsetminus D(\Gamma)$ has codimension ≥ 2 (provided $\dim D(\Gamma) \geq 2$, of course).

(b) When $D(\Gamma)$ is compact, the theorem follows from the Kodaira embedding theorem (Wells 1980). Nadel and Tsuji (J Diffl Geom, 1988, 503-512, Theorem 3.1) extended this to $D(\Gamma)$ with boundary of dimension 0, and Mok, N., and Zhong, Jia Qing, (Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature. Ann. of Math. (2) 129 (1989), no. 3, 427–470) give an alternative prove the Baily-Borel theorem, but without some of the information on the boundary.

An algebraic variety $D(\Gamma)$ arising as in the theorem is called a ***locally symmetric variety***, or an ***arithmetic locally symmetric variety***, or an ***arithmetic variety*** (but not a Shimura variety).

The theorem of Borel

THEOREM 3.13 (BOREL). *Let $D(\Gamma)$ and $D(\Gamma)^*$ be as in (3.11). Let V be a nonsingular quasi-projective variety over $\mathbb{C}$. Then every holomorphic map $f\colon V^{an} \to D(\Gamma)^{an}$ is regular, and extends to a regular map $f^*\colon M^* \to D(\Gamma)^*$.*

The key step in Borel's proof is the following result:

LEMMA 3.14. *Let $D_1^\times$ be the punctured disk $\{z \mid 0 < |z| < 1\}$. Then every holomorphic map $D_1^{\times r} \times D_1^s \to D(\Gamma)$ extends to a holomorphic map $D_1^{r+s} \to D(\Gamma)^*$ (unfortunately $D(\Gamma)^*$ is not a complex manifold (it has singularities), but rather a complex analytic variety, which means it can be locally defined as the zero set of a collection of power series).*

The original result of this kind is the big Picard theorem, which, interestingly, was first proved using elliptic modular functions. Recall that the theorem says that if a function f has an essential singularity at a point $p \in \mathbb{C}$, then on any open disk containing p, f takes every complex value except possibly one. Therefore, if a holomorphic function f on $D_1^\times$ omits two values in $\mathbb{C}$, then it has at worst a pole at 0, and so extends to a holomorphic function $D_1 \to \mathbb{P}^1(\mathbb{C})$. This can be restated as follows: every holomorphic function from $D_1^\times$ to $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{3 \text{ points}\}$ extends to a holomorphic function from D_1 to the natural compactification $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{3 \text{ points}\}$. Over the decades, there were various generalizations of this theorem. For example, Kwack (1969) replaced $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{3 \text{ points}\}$ with a more general class of spaces. Borel (1972) verified Kwack's theorem applied to $D(\Gamma) \subset D(\Gamma)^*$, and extended the result to maps from a product $D_1^{\times r} \times D_1^s$.

Using the lemma, we can prove the theorem. According Hironaka's [Fields medal] theorem on the resolution of singularities (Hironaka 1964 Annals), we can realize V as an open subvariety of a projective nonsingular variety V^* such that $V^* \smallsetminus V$ is a divisor with normal crossings. This means that locally for the complex topology the inclusion $V \hookrightarrow V^*$ is of the form $D_1^{\times r} \times D_1^s \hookrightarrow D_1^{r+s}$. Therefore, the lemma shows that $f\colon V^{\mathrm{an}} \to D(\Gamma)^{\mathrm{an}}$ extends to a holomorphic map $V^{*\mathrm{an}} \to D(\Gamma)^*$, which is regular by Chow's theorem (3.10).

COROLLARY 3.15. *The structure of an algebraic variety on $D(\Gamma)$ is unique.*

PROOF. Apply (3.14) to the identity map. □

The compactification $D(\Gamma) \hookrightarrow D(\Gamma)^*$ has the following property: for any compactification $D(\Gamma) \to D(\Gamma)^\dagger$ with $D(\Gamma)^\dagger \smallsetminus D(\Gamma)$ a divisor with normal crossings, there is a unique regular map $D(\Gamma)^\dagger \to D(\Gamma)^*$ making

$$\begin{array}{ccc} & & D(\Gamma)^\dagger \\ & \nearrow & \downarrow \\ D(\Gamma) & & \\ & \searrow & \downarrow \\ & & D(\Gamma)^* \end{array}$$

commute. For this reason, $D(\Gamma) \hookrightarrow D(\Gamma)^*$ is often called the ***minimal*** compactification. Other names: ***standard***, ***Satake-Baily-Borel***, ***Baily-Borel.***

Finiteness of the automorphism group

PROPOSITION 3.16. *The automorphism group of the quotient of a hermitian symmetric domain by a neat arithmetic group is finite.*

PROOF. As Γ is a neat arithmetic subgroup of $\mathrm{Aut}(D)$, D is the universal covering space of $\Gamma\backslash D$ and Γ is the group of covering transformations (Greenberg 1967, 5.8). An automorphism $\alpha\colon \Gamma\backslash D \to \Gamma\backslash D$ lifts to an automorphism $\tilde{\alpha}\colon D \to D$. For any $\gamma \in \Gamma$, $\tilde{\alpha}\gamma\tilde{\alpha}^{-1}$ is a covering transformation, and so lies in Γ. Conversely, an $\tilde{\alpha}$ in the normalizer N of Γ defines an automorphism of $\Gamma\backslash D$. Thus, $\mathrm{Aut}(\Gamma\backslash D) = N/\Gamma$. Because Γ is closed in $\mathrm{Aut}(D)$ and is countable, so also is $N \subset \mathrm{Aut}(\Gamma)$. If N were not discrete, every finite subset of it would have empty interior, and since N is the countable union of such sets, this would violate the Baire category theorem (Kelley 1955, pp200–201). Because the quotient of $\mathrm{Aut}(D)$ by Γ has finite measure, this implies that Γ has finite index in N. (See also Margulis 1991, II 6.3.)

Alternatively, there is a geometric proof. According to Mumford 1977, Proposition 4.2, such a quotient is an algebraic variety of logarithmic general type, which implies that its automorphism group is finite (Iitaka 1982, 11.12). □

4 Connected Shimura varieties

Congruence subgroups

Let G be a semisimple algebraic group over $\mathbb{Q}$. Choose a faithful representation $\rho\colon G \to \mathrm{GL}(V)$ and a lattice Λ in V, and define

$$\Gamma(N) = \{g \in G(\mathbb{Q}) \mid \rho(g)\Lambda \subset \Lambda, \quad (\rho(g)-1)\Lambda \subset N\Lambda\}.$$

For example, if $G = \mathrm{GL}_2$, $V = \mathbb{Q}^2$ with its natural action of G, and $\Lambda = \mathbb{Z}^2$, then

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, d \equiv 1, \quad b, c \equiv 0 \mod N \right\}.$$

A ***congruence subgroup of*** $G(\mathbb{Q})$ is any subgroup containing some $\Gamma(N)$ as a subgroup of finite index. It is not difficult to see that this definition is independent of the choice of ρ and Λ, but we shall give a more natural definition of congruence subgroup shortly.

With this terminology, an arithmetic subgroup of $G(\mathbb{Q})$ is any subgroup commensurable with $\Gamma(1)$. The congruence subgroup problem for G asks whether every arithmetic subgroup of $G(\mathbb{Q})$ is congruence, i.e., contains some $\Gamma(N)$. For some simply connected groups (e.g., SL_n for $n \geq 3$ or Sp_{2n} for $n \geq 2$ Bass, Milnor, and Serre) the answer is yes, but SL_2 and all nonsimply connected group have many noncongruence arithmetic subgroups (for a discussion of the problem, see Platonov & Rapinchuk 1994, section 9.5).

In contrast to arithmetic subgroups, the image of a congruence subgroup under an isogeny of algebraic groups need not be a congruence subgroup.

We define $\mathbb{A}_f$, the ***ring of finite adèles*** to be the restricted topological product

$$\mathbb{A}_f = \prod (\mathbb{Q}_\ell\colon \mathbb{Z}_\ell)$$

where ℓ runs over the finite primes of ℓ (i.e., we omit the factor $\mathbb{R}$). Thus, $\mathbb{A}_f$ is the subring of $\prod \mathbb{Q}_\ell$ consisting of the (a_ℓ) such that $a_\ell \in \mathbb{Z}_\ell$ for almost all ℓ, and it is endowed with the topology for which $\prod \mathbb{Z}_\ell$ is open and has the product topology. Similarly, for an algebraic group G over $\mathbb{Q}$, we define $G(\mathbb{A}_f)$ to be the restricted product

$$G(\mathbb{A}_f) = \prod (G(\mathbb{Q}_\ell)\colon G(\mathbb{Z}_\ell)).$$

To define $G(\mathbb{Z}_\ell)$, we choose a representation $G \hookrightarrow \mathrm{GL}(V)$ and a lattice Λ and let $G(\mathbb{Z}_\ell)$ be the stabilizer of $\Lambda \otimes \mathbb{Z}_\ell$ in $G(\mathbb{Q}_\ell)$. The group $G(\mathbb{Z}_\ell)$ depends on the choice of V and Λ, but for any two choices, $G(\mathbb{Z}_\ell)$ will coincide for almost all ℓ, and so $G(\mathbb{A}_f)$ is independent of the choice. Alternatively (equivalently), extend G to a group scheme $\mathcal{G}$ over an open subset U of $\operatorname{Spec}\mathbb{Z}$, and define $G(\mathbb{Z}_\ell) = \mathcal{G}(\mathbb{Z}_\ell)$ for $\ell \in U$. If $\mathcal{G}'/U'$ is a second such group scheme, then $\mathcal{G}' = \mathcal{G}$ over some open subset of $U \cap U'$, and so $\mathcal{G}'(\mathbb{Z}_\ell) = \mathcal{G}(\mathbb{Z}_\ell)$ for almost all ℓ.

Now, the congruence subgroups of $G(\mathbb{Q})$ are those of the form $G(\mathbb{Q}) \cap K$ with K a compact open subgroup of $G(\mathbb{A}_f)$.

Summary: $\{$congruence subgroups$\} \subset \{$arithmetic subgroups$\} \subset \{$lattices$\}$.

Definition of a connected Shimura variety

Recall that $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$ in the real topology. We set $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. For example, $\mathrm{GL}_2(\mathbb{Q})^+$ is the group of 2×2 matrices with rational coefficients having positive determinant.

4.1. Let G be a semisimple algebraic group over $\mathbb{Q}$, let D be a hermitian symmetric domain, and let $G(\mathbb{R})^+ \times D \to D$ be a transitive action such that the stabilizer of any point of D is compact. It is convenient to assume also that G has no connected normal subgroup H (over $\mathbb{Q}$) such that $H(\mathbb{R})$ is compact. (If G did, then $H(\mathbb{R})^+$ would act trivially on D, and so we could simply replace G with G/H).

The action of $G(\mathbb{R})^+$ on D factors through the quotients[7]

$$G(\mathbb{R})^+ \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})^+ \twoheadrightarrow \mathrm{Aut}(D)^+. \tag{24}$$

The kernel of the first map is finite, and the kernel of the second is a compact factor of $G^{\mathrm{ad}}(\mathbb{R})^+$.

Define $\mathrm{Sh}^\circ(G,D)$ to be the set of all locally symmetric varieties $D(\Gamma) = \Gamma\backslash D$ with Γ a torsion-free arithmetic subgroup of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$ (i.e., whose inverse image in $G(\mathbb{Q})^+$ is a congruence subgroup). The $D(\Gamma)$ form an inverse system,

$$D(\Gamma) \leftarrow D(\Gamma') \text{ if } \Gamma \supset \Gamma'.$$

Note that the image of Γ in $\mathrm{Aut}(D)^+$ is an arithmetic group (directly from the definition). Therefore, the theorems of Baily-Borel and Borel apply and show that the $D(\Gamma)$ are algebraic varieties and the maps $D(\Gamma) \leftarrow D(\Gamma')$ are regular. Moreover, there is an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on the system: $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$ defines a holomorphic map $g\colon D \to D$, and hence a map

$$\Gamma\backslash D \to g\Gamma g^{-1}\backslash D.$$

This is holomorphic (see 3.2), and hence regular (3.13).

EXAMPLE 4.2. (a) $G = \mathrm{SL}_2$, $D = \mathcal{H}_1$. Then $\mathrm{Sh}^\circ(G,D)$ is the family of elliptic modular curves $\Gamma\backslash D$ with Γ a torsion-free arithmetic in $\mathrm{PGL}_2(\mathbb{R})^+$ and containing the image of $\Gamma(N)$ for some N.

(b) $G = \mathrm{PGL}_2$, $D = \mathcal{H}_1$. The same as (a), except now the Γ are required to be congruence subgroups of $\mathrm{PGL}_2(\mathbb{Q})$ — there are many fewer of these.

(c) Let B be a quaternion algebra over a totally real field F. Then

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong \textstyle\prod_{v\colon F \hookrightarrow \mathbb{R}} B \otimes_{F,v} \mathbb{R}$$

[7] It is probably helpful if I note that $\mathrm{Aut}(D)^+$ is the same whether we consider D as a riemannian manifold (i.e., forget the complex structure), a complex manifold (i.e., forget the hermitian structure), or as a hermitian complex manifold (see Baily & Borel 1966, Appendix). This is not true without the + — there may exist antiholomorphic isometries.

and each $B\otimes_{F,v}\mathbb{R}$ is isomorphic to $\mathbb{H}$ (the usual quaternions) or $M_2(\mathbb{R})$. Let G be the semisimple algebraic group over $\mathbb{Q}$ such that

$$G(\mathbb{Q})=\mathrm{Ker}(\mathrm{Nm}\colon B^\times\to F^\times),$$

and similarly for any $\mathbb{Q}$-algebra R. Then

$$G(\mathbb{R})=\mathbb{H}^{\times 1}\times\cdots\times\mathbb{H}^{\times 1}\times\mathrm{SL}_2(\mathbb{R})\times\cdots\times\mathrm{SL}_2(\mathbb{R}).$$

Assume at least one $\mathrm{SL}_2(\mathbb{R})$ occurs, and let D be a product of copies of $\mathcal{H}_1$, one for each copy of $\mathrm{SL}_2(\mathbb{R})$. There is a natural action of $G(\mathbb{R})$ on D which satisfies the conditions of (4.1), and hence we get a connected Shimura variety. Note that, in this case, (24) becomes

$$\mathbb{H}^{\times 1}\times\cdots\times\mathrm{SL}_2(\mathbb{R})\times\cdots\to\mathbb{H}^{\times 1}/\pm 1\times\cdots\times\mathrm{SL}_2(\mathbb{R})/\pm I\to\mathrm{SL}_2(\mathbb{R})/\pm I\times\cdots.$$

In this case, $D(\Gamma)$ has dimension equal to the number of copies of $M_2(\mathbb{R})$ in the decomposition of $B\otimes_{\mathbb{Q}}\mathbb{R}$.

REMARK 4.3. Let G be a connected reductive group over $\mathbb{Q}$, and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Then $\Gamma\backslash G(\mathbb{R})$ is compact if and only if $G(\mathbb{Q})$ has no unipotent elements $\neq 1$ (Borel 1969, 8.7). The intuitive reason for this is the rational unipotent elements correpond to cusps (in the case of SL_2 acting on $\mathcal{H}_1$), so no rational unipotent elements means no cusps.

In the above example, if $B=M_2(F)$ it has unipotent elements, e.g., $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and so $\Gamma\backslash G(\mathbb{R})$ is not compact — in this case we get the ***Hilbert modular varieties***. This implies that the $D(\Gamma)$ are not compact. On the other hand, if B is a division algebra, $G(\mathbb{Q})$ has no unipotent elements (because otherwise B would have a nilpotent element). Thus, in this case the $D(\Gamma)$ are compact (as manifolds, hence they are projective as algebraic varieties).

Strong approximation theorem

THEOREM 4.4 (STRONG APPROXIMATION). *If G is a simply connected semisimple algebraic group over $\mathbb{Q}$ with no $\mathbb{Q}$-factor H such that such that $H(\mathbb{R})$ is compact, then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$.*

For the proof, see Platonov & Rapinchuk 1994, Theorem 7.12, p427. Note that, because $G(\mathbb{Z})$ is discrete in $G(\mathbb{R})$ (3.5), if $G(\mathbb{R})$ is compact, then $G(\mathbb{Z})$ is finite, and so $G(\mathbb{Z})$ is not dense in $G(\hat{\mathbb{Z}})$, which implies that $G(\mathbb{Q})$ is not dense in $G(\mathbb{A}_f)$.

Adèlic description of $D(\Gamma)$

Assume G to be simply connected.

PROPOSITION 4.5. *Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let*

$$\Gamma=K\cap G(\mathbb{Q})$$

be the corresponding congruence subgroup of $G(\mathbb{Q})$. *The map* $x \mapsto [x,1]$ *defines a bijection*

$$\Gamma\backslash X^{+} \cong G(\mathbb{Q})\backslash X^{+} \times G(\mathbb{A}_f)/K. \tag{25}$$

Here $G(\mathbb{Q})$ *acts on both* X^{+} *and* $G(\mathbb{A}_f)$ *on the left, and* K *acts on* $G(\mathbb{A}_f)$ *on the right:*

$$q \cdot (x,a) \cdot k = (qx, qak), \quad q \in G(\mathbb{Q}), \quad x \in X^{+}, \quad a \in G(\mathbb{A}_f), \quad k \in K.$$

When we endow X^{+} *with its usual topology and* $G(\mathbb{A}_f)$ *with the adèlic topology (equivalently, the discrete topology), this becomes a homeomorphism.*

PROOF. Because K is open, $G(\mathbb{A}_f) = G(\mathbb{Q})\cdot K$. Therefore, every element of $G(\mathbb{Q})\backslash X^{+} \times G(\mathbb{A}_f)/K$ is represented by an element of the form $[x,1]$. By definition, $[x,1] = [x',1]$ if and only if there exist $q \in G(\mathbb{Q})$ and $k \in K$ such that $x' = qx$, $1 = qk$. The second equation implies that $q = k^{-1} \in \Gamma$, and so $[x,1] = [x',1]$ if and only if x and x' represent the same element in $\Gamma\backslash X^{+}$.

Consider

$$\begin{array}{ccc} X^{+} & \xrightarrow{x\mapsto(x,[1])} & X^{+} \times (G(\mathbb{A}_f)/K) \\ \downarrow & & \downarrow \\ \Gamma\backslash X^{+} & \xrightarrow{[x]\mapsto[x,1]} & G(\mathbb{Q})\backslash X^{+} \times G(\mathbb{A}_f)/K. \end{array}$$

As K is open, $G(\mathbb{A}_f)/K$ is discrete, and so the upper map is a homeomorphism of X^{+} onto its image, which is open. It follows easily that the lower map is a homeomorphism. □

ASIDE 4.6. (a) What happens when we pass to the inverse limit over Γ? There is a map

$$X^{+} \to \varprojlim \Gamma\backslash X^{+}$$

which is injective because $\cap\Gamma = \{1\}$. Is the map surjective? The example

$$\mathbb{Z} \to \varprojlim \mathbb{Z}/m\mathbb{Z} = \hat{\mathbb{Z}}$$

is not encouraging — it suggests $\varprojlim \Gamma\backslash X^{+}$ might be some sort of completion of X^{+} relative to the Γ's. This is correct. In fact, when we pass to the limit on the right in (3), we get the obvious answer, namely,

$$\varprojlim{}_K G(\mathbb{Q})\backslash X^{+} \times G(\mathbb{A}_f)/K = G(\mathbb{Q})\backslash X^{+} \times \mathrm{SL}_2(\mathbb{A}_f).$$

Why the difference? Well, given an inverse system $(G_i)_{i\in K}$ of groups acting on an inverse system $(S_i)_{i\in I}$ of topological spaces, there is always a canonical map

$$\varprojlim G_i\backslash \varprojlim S_i \to \varprojlim (G_i\backslash S_i)$$

and it is known that, under certain hypotheses, the map is an isomorphism (Bourbaki 1989, III §7). The system on the right in (3) satisfies the hypotheses; that on the left doesn't.

(b) Why replace the single coset space on the left with the more complicated double coset space on the right? One reason is that it makes transparent that there is an action of $G(\mathbb{A}_f)$ on the inverse system $(\Gamma\backslash X^+)_\Gamma$ when G is simply connected, and hence, for example, an action of $G(\mathbb{A}_f)$ on

$$\varinjlim H^1(\Gamma\backslash X^+, \mathbb{Q}).$$

Another reason will be seen presently — we use double cosets to define Shimura varieties. Double coset spaces are pervasive in work on the Langlands program.

(c) The inverse limit of the $D(\Gamma)$ exists as a scheme — it is even noetherian and regular. However, there seems to me no advantage in working with it rather than the inverse system.

Corrections:

The arrows in the bottom row of (16) point in the wrong direction.

Some of the discussion leading up to Theorem 1.19 may be a bit oversimplified. For example, in (1.16) it is probably better to say that the hypothesis implies that the curvature tensor is invariant under parallel translation, and hence so also the sectional curvature. See Deligne 1973 or Wolf 1984 for more details.