1. Ramsey theory and Banach spaces, Fields Institute, Fall 2002. Problem Set #1
September 9, 2002

Hand in solutions to problems #3, #5, #7, #8 and #10; use the other problems as a warm-up. Due date is **September 23, 2002**.

Exercise 1. (1) *Prove that every sequence* $\{x_n\}_{n=1}^\infty$ *or reals contains an infinite monotonic subsequence.*
(2) *Prove that every infinite partial ordering contains an infinite chain or an infinite antichain.*
(3) *Prove that every linear ordering contains an isomorphic copy of* $(\mathbb{N},<)$ *or an isomorphic copy of* $(\mathbb{N},>)$.

Exercise 2. *Assume* I_i $(i \in \mathbb{N})$ *are closed nonempty intervals included in* $[0,1]$.
(a) *Prove that the following are equivalent:*
(1) *For every infinite* $A \subseteq \mathbb{N}$ *there are* $i \neq j$ *in* A *such that* $I_i \cap I_j \neq \emptyset$.
(2) *There is an infinite* $A \subseteq \mathbb{N}$ *such that* $I_i \cap I_j \neq \emptyset$ *for all* i *and* j *in* A.
(3) *There is an infinite* $A \subseteq \mathbb{N}$ *such that* $\bigcap_{i\in s} I_i$ *for every finite* $s \subseteq A$.
(4) *There is* $x \in [0,1]$ *such that* $\{i : x \in I_i\}$ *is infinite.*
(b) *Now assume that* I_i $(i \in \mathbb{N})$ *are arbitrary subsets of* $\mathbb{R}$. *What can you say about the relationship between (1)–(4)? Provide a counterexample or a proof for each of the implications (n)⇒ (n+1)* $(n = 1, 2, 3)$.

Exercise 3. (a) *Assume* G *is an infinite group. Prove that there is an infinite* $A \subseteq G$ *such that for all* $\{x, y, z\} \subseteq A$ *we have* $xy = z$ *if and only if* $x = y = z$ *and* x *is an idempotent.*
(b) *Does (a) remain true if* G *is assumed to be a semigroup instead of a group? Justify your answer.*

Exercise 4. *Assume* G *is a group and that* a_iH_i $(i < n)$ *are its cosets such that* $G = \bigcup_{i=0}^{n-1} a_iH_i$. *Let* $s = \{i < n : [G : H_i]$ *is finite*$\}$. *Prove that* $G = \bigcup_{i=0}^{n-1} a_iH_i$.

Exercise 5. *Assume* $\{x_n\}_{n=1}^\infty$ *is a sequence in* $[0,1]$. *Find a subsequence* $\{x_{n_k}\}_{k=1}^\infty$ *such that both* $\lim_{k\to\infty} x_{n_k}$ *and* $\lim_{k\to\infty} x_{n_{k+1}-n_k}$ *exist.*

Exercise 6. *Assume* $k \in \mathbb{N}$ *and* $f\colon \mathbb{N}^2 \to k$. *Prove that there is an infinite* $A \subseteq \mathbb{N}$ *and* $\{i_1, i_2, i_3\} \subseteq k$ *such that for all* $\{m, n\} \subseteq A$ *we have*

$$f(m,n) = \begin{cases} i_1, & \text{if } m < n \\ i_2, & \text{if } m = n \\ i_3, & \text{if } m > n \end{cases}$$

Maps $f\colon \mathbb{N}^d \to Y$ and $g\colon \mathbb{N}^d \to Y$ are *isomorphic on* A (for some $A \in [\mathbb{N}]^\omega$) if for all $\{s,t\} \subseteq A^d$ we have $f(s) = f(t)$ if and only if $g(s) = g(t)$. A family $\mathcal{B}$ of functions forms a *basis* for $\mathcal{H}_d = \{f : \mathrm{dom}(f) = \mathbb{N}^d\}$ if for every $f \in \mathcal{H}_d$ there exist $g \in \mathcal{B}$ and $A \in [\mathbb{N}]^\omega$ such that f is isomorphic to g on A.

Exercise 7. (1) *Prove that for every* $d \in \mathbb{N}$ *there is a finite basis* $\mathcal{B}_d$ *for* $\mathcal{H}_d$.
(2) *Find the minimal possible size of* $\mathcal{B}_d$ *for* $d = 2, 3, \dots$.

Exercise 8. *Assume that* $k \in \mathbb{N}$ *and* I_n *is a set of size* k *for every* $n \in \mathbb{N}$. *Prove that there is an infinite* $A \subseteq \mathbb{N}$ *and a finite* R *such that* $I_m \cap I_n = R$ *for all* m *and* n *in* A.

Recall that $[\mathbb{N}]^\omega = \{A \subseteq \mathbb{N} : A \text{ is infinite}\}$. A family $\mathcal{F} \subseteq [\mathbb{N}]^\omega$ is *dense* if for every $A \in [\mathbb{N}]^\omega$ there is $B \subseteq A$ in $\mathcal{F}$. For $A \in [\mathbb{N}]^\omega$ let

$$A/n = A \setminus (n+1).$$

Lemma 9 (Diagonalization Lemma). *If each* $\mathcal{F}_n \subseteq [\mathbb{N}]^\omega$ $(n \in \mathbb{N})$ *is dense, then there is* $A \in [\mathbb{N}]^\omega$ *such that* $A/n \in \bigcap_{m\leq n} \mathcal{F}_m$ *for every* $n \in A$.

Exercise 10. *Prove or disprove the following strengthening of Diagonalization Lemma:*
If each $\mathcal{F}_n \subseteq [\mathbb{N}]^\omega$ $(n \in \mathbb{N})$ *is dense, then there is* $A \in [\mathbb{N}]^\omega$ *such that* $A/n \in \bigcap_{m\leq n} \mathcal{F}_m$ *for every* $n \in \mathbb{N}$.

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