Some set-theoretic problems from convexity theory

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Convexity

- Let V be a real-linear space. A set $C \subseteq V$ is convex if for all $x, y \in S$ the line segment $[x, y] := \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$ is contained in S.
- the convex hull $\mathrm{conv}(X)$ of a set $X \subseteq V$ is the intersection of all convex sets containing X, or, equivalently, obtained from X by repeatedly adding all $[x, y]$ in ω steps. $\mathrm{conv}(X) = \bigcup\{\mathrm{conv}(Y) : Y \in [X]^{<\aleph_0}\}$.
- Caratheodory's theorem: if $d < \infty$ and $X \subseteq \mathbb{R}^d$ then $y \in \mathrm{conv}(X)$ iff there is some $A \in [X]^{\leq(d+1)}$ so that $y \in \mathrm{conv}(A)$.
- A set $X \subseteq S \subseteq V$ is defected in S if $\mathrm{conv}(X) \not\subseteq S$.

Convex covers

- Given $S \subseteq V$, let $I(S)$ be the ideal generated over S by all convex subsets of S. The covering number of this ideal, $\mathrm{Cov}(I(S))$ is the number of convex subsets of S required to cover S, called also the convexity number of S and sometimes written $\gamma(S)$.
- $\gamma(S)$ is the chromatic number of the hypergraph (S, E) where E is the collection of finite defected subsets of S.
- If $\gamma(S) \leq \aleph_0$ we say that S is countably convex.
- Quite a few set-theoretic problem result from studying the structure of either countably or uncountably convex sets in Banach spaces.

Separating the countable from the uncountable part

Work now in a second countable topological vector space V. Given a set $S \subseteq V$, let $A = \bigcup\{S \cap u : u \text{ basic open and } \gamma(S \cap u) \leq \aleph_0\}$. Let $B = S \setminus A$. So:

- $\gamma(A) \leq \aleph_0$
- A is open; so B is closed.
- B is perfect.
- $\gamma(S) > \aleph_0 \iff B \neq \emptyset$

Effective convexity numbers

Depending on the dimension and on the descriptive complexity of the set S, there are two effective ways to compute the convexity number of S.

- A subset $P \subseteq S$ is k-clique ($k \geq 2$) if all k-subsets of P are defected in S. A perfect k-clique $P \subseteq S$ is an effective evidence that $\gamma(S) = 2^{\aleph_0}$.
- On the other hand there is a rank function $\rho_S(x)$ which measures the convex complexity of a point $x \in S$, and in some cases provides countable convex covers effectively (Kojman 2000).

Part I: Countable convexity

- The rank function: for every ordinal α,

$$\rho_S(x) \geq \alpha \iff (\forall \text{ open } u \ni x)(\forall \beta < \alpha)$$
$$(\exists \text{ defected } Y \subseteq u) \bigwedge_{y \in Y} \rho_S(y) \geq \beta$$

- A point has rank $\geq \alpha$ if it is a limit of defected configurations of points of arbitrarily large rank below α.
- There is an ordinal $\alpha(S) < \omega_1$ so that for all $x \in S$, if $\rho_S(x) > \alpha(S)$ then $\rho_S(x) = \infty$

- There is an effective way to cover $\{x \in S : \rho_S(x) \leq \alpha\}$ by countably many convex sets: for every point of rank $\beta \leq \alpha$, there is an open neighborhood in which the convex hull of all points of the same rank is contained in S.
- Call $K(S) = \{x \in S : \rho_S(x) = \infty\}$ the convexity radical of S.
- It can happen that $K(S) \neq \emptyset$ but $\gamma(S) \leq \aleph_0$ in an F_σ set. Let S be the union of all vertical lines at rational distance from the y-axis.
- A closed subset S of a Polish vector space is countably convex iff $K(S) = \emptyset$.

Application: The unit sphere in $C(K)$

Let K be a compact metric space, and let $C(K)$ be the Banach space of all continuous real functions on K, with the sup norm. Let $S(K) = \{f \in C(K) : \|f\| = 1\}$ denote the unit sphere in $C(K)$.

If K is uncountable, $S(K)$ is not countably convex.

Theorem. If K_1, K_2 are compact metric spaces and $\rho(S(K_1)) = \rho(S(K_2)) < \infty$ then $K_1 \cong K_2$.

G_δ sets

The following hold for G_δ sets (Fonf-Kojman 2001):

- in a finite dimensional G_δ set S, the radical $K(S)$ is always nowhere-dense in S.
- In dimension $d \leq 3$, a countably convex G_α set cannot contain a dense in itself clique.
- In dimension $d \geq 4$ there is a countably convex G_δ set with a dense in itself 2-clique.
- In every infinite dimensional Banach space there is a countably convex G_δ set S which contains a 2-clique which is dense in itself and in S.

$\mathbb{R}^4$

Let

$$L(t) = (t, t^2, t^3, t^4)$$

and let $L = \{L(t) : t \in [0, 1]\}$
Let S be the convex hull of L from which we remove, for any two rational $t_1, t_2 \in \mathbb{Q} \cap [0, 1]$, the mid-point $(L(t_1) + L(t_2))/2$.
S is a G_δ set and $\{L(t) : t \in \mathbb{Q} \cap (0, 1)\}$ is a dense in itself 2-clique. Why is S countably convex?
For $t_1, t_2 \in [0, 1]$,
$(x - t_1)^2(x - t_2)^2 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.
Let $T : \mathbb{R}^4 \to \mathbb{R}$ be defined by
$T(v_1, v_2, v_3, v_4) = a_0 + \sum_{i=1}^4 a_iv_i$. For all $t \in [0, 1]$,
$T(L(t)) = T(t, t^2, t^3, t^4) = (t - t_1)^2(t - t_2)^2 \geq 0$. Thus, $L(t_1), L(t_2)$ are on a supporting hyperplane.

$\mathbb{R}^3$

Suppose $S = \bigcup C_n$ is a G_δ set and $P \subseteq S$ is a dense in itself 2-clique. Then one of the C_n in somewhere dense in the closure of P.
Get: a dense in itself subset on the boundary of a convex subset, with any two points connected via the boundary.
Then there is a plane which contains a dense in itself subset of P. Why? Because in $\mathbb{R}^3$, any simplicial polytope with 5 vertices or more contains an inner diagonal.
Similarly, for k-cliques with $k > 2$, use the fact: every simplicial polytope in R^3 with $4k + 1$ vertices or more has an inner polytope with $d + 1$ vertices.

Part II: Uncountable convexity

If $S \subseteq V$ is closed, then $K(S) = \emptyset \iff \gamma(S) \leq \aleph_0$, because the the closure of a convex set is conves; therefore the intersection of a convex subset of S with $K(S)$ is nowhere dense.

What meager ideal are realized as convexity ideals in which dimensions? Can we learn more about meager ideals in general from those examples?

Digression: Covering by Meager ideals

- Let $\mathcal{M}(X)$ denote the meager ideal over a perfect Polish space X, and let $\mathrm{Cov}(\mathcal{M}(X))$ denote the number of members $\mathcal{M}(X)$ necessary to cover X (which is always uncountable by the Baire theorem).
- A meager ideal is any ideal $I \subseteq M(X)$ for a perfect Polish X.
- Goldstern-Shelah: there is a model of set theory in which $\aleph_1$ different uncountable cardinals are realized as the covering numbers of simply defined meager ideals. So the landscape is complicated.

Largeness of $\mathcal{M}$

Meager sets can be big in other senses:

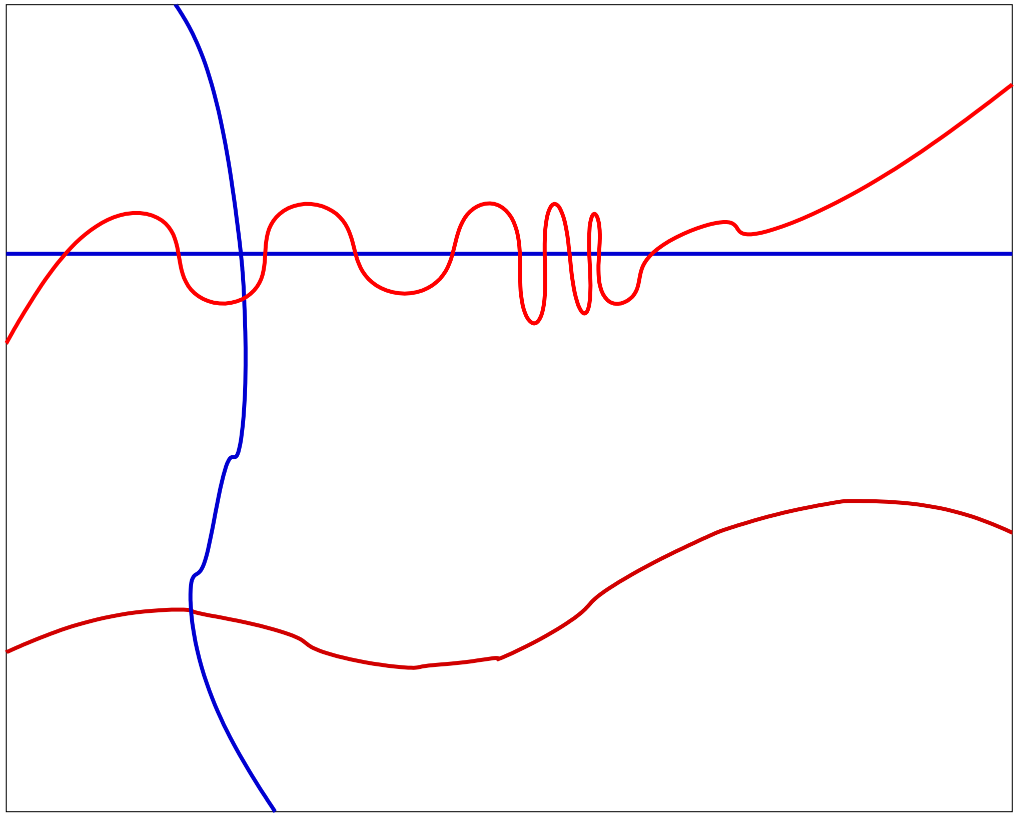
- $\mathbb{R} = A \cup B$, A meager, B of Lebesgue measure 0.
- In Forcing terminology: adding a random real makes the set of ground model reals meager.
- Thus, after adding $\aleph_1$ random reals, $\mathbb{R}$ is covered by $\aleph_1$ meager sets: It is consistent that $|\mathbb{R}| = \aleph_{100}$ and $\mathrm{Cov}(\mathcal{M}) = \aleph_1$

***Big* and *trivial* meager ideals**

- The consistency of $\mathrm{Cov}(\mathcal{M}) \ll 2^{\aleph_0}$ will be taken as a definition that $\mathcal{M}$ is a big meager ideal; similarly, any meager I is big if it is consistent that $\mathrm{Cov}(I) \ll 2^{\aleph_0}$.
- On the other hand, a meager ideal $I \subseteq \mathcal{M}$ is called trivial if $ZFC \vdash \mathrm{Cov}(I) = 2^{\aleph_0}$.
- Example: the ideal of countable subsets of $\mathbb{R}$ is trivial.

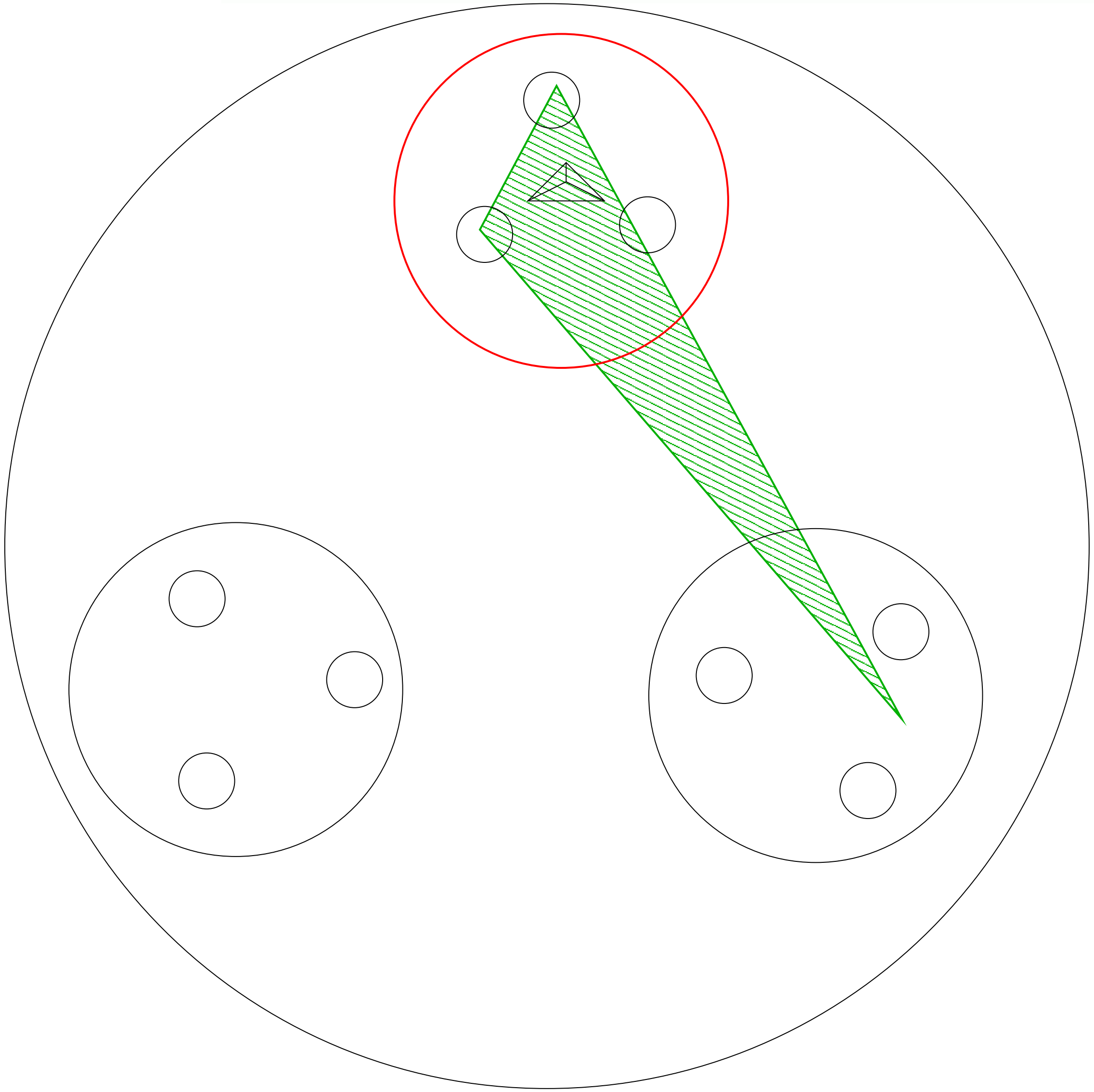
Another example

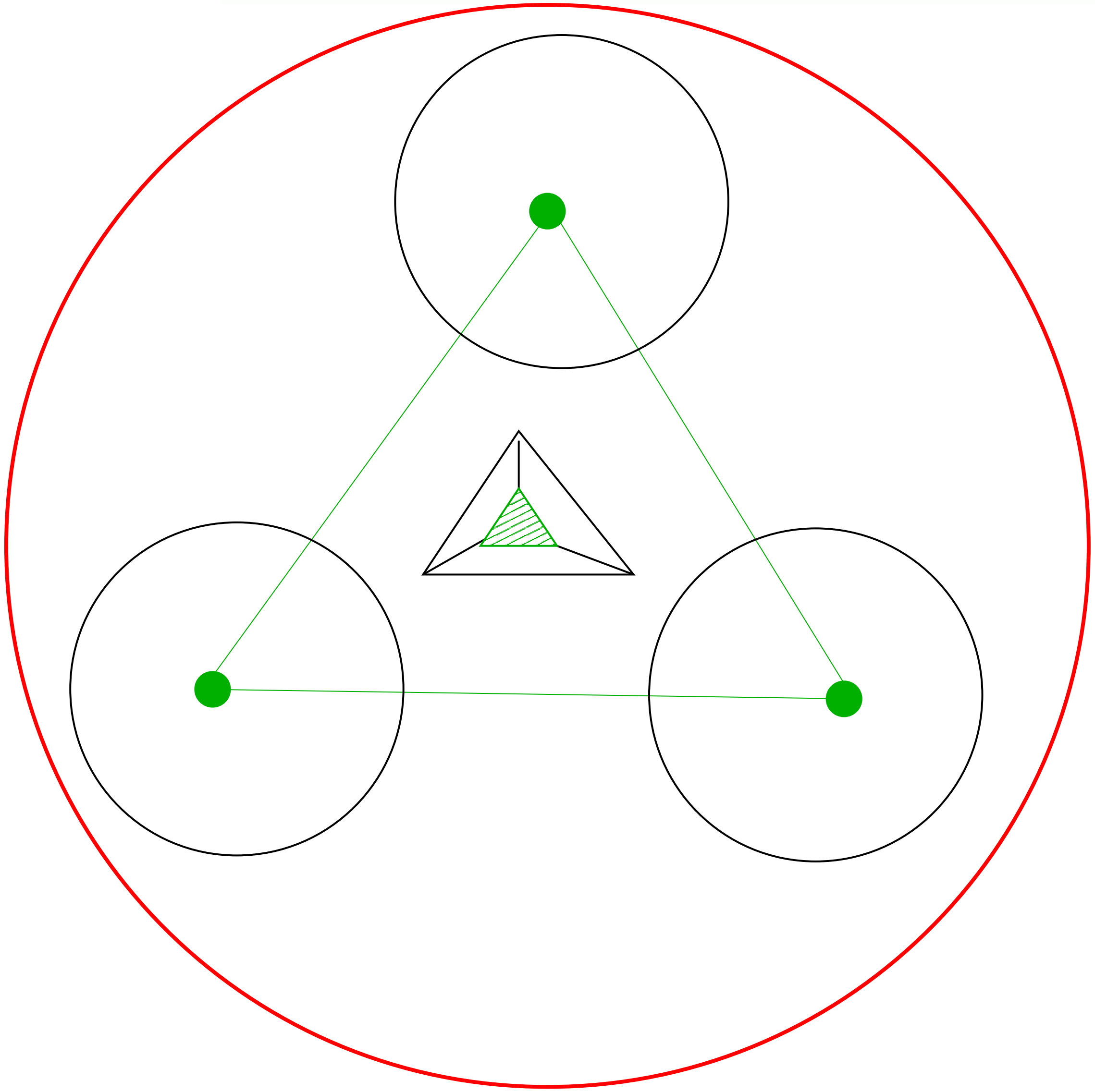
The ideal generated over $\mathbb{R}^2$ by graphs of real-analytic functions and their inverses.



Meager ideals from convexity

For every closed uncountably convex $S \subseteq R^d$, the ideal $I(S)$ on $K(S)$ is meager.
Example. The ideal generated by 2-branching perfect subtrees of 3^ω is big and is isomorphic to the convexity ideal of the following set:





$\mathbb{R}^n$ *and the dimension conjecture*

Theorem. (Geschke-Kojman 2002) for all $n > 2$ there are closed sets $S_1, S_2, S_{n-1} \subseteq \mathbb{R}^n$ so that for every sequence of cardinals $\kappa_1 > \kappa_2 > \cdots > \kappa_{n-1}$, each with uncountable cofinality, it is consistent that $\gamma(S_i) = \kappa_i$.

Conjecture. For every n, it is consistent that n different uncountable cardinals can be realized as convexity numbers of closed subsets of $\mathbb{R}^n$, but not more.

$\mathbb{R}^2$

(Geschke-Kojman-Kubis-Schipperus 200?) The dimension conjecture holds in $\mathbb{R}^2$! For every close $S \subseteq \mathbb{R}^2$, either S contains a perfect clique, or else $\gamma(S)$ is equal to the homogeneity number $\mathfrak{hm}(c)$ of some continuous pair coloring (geometric proof).

A closed $S \subseteq \mathbb{R}^2$ contains a perfect 3-clique iff in the Sacks extension its convexity number remains continuuum.

In fact, the nontrivial convexity ideals of closed sets in $\mathbb{R}^2$ are a new type of very small — yet nontrivial — meager ideals.
Is $I(c_{\max})$ realizable as a convexity ideal of a closed set in $\mathbb{R}^2$?

More connections and more problems

- Is there a smallest nontrivial meager ideal?
- Is $I(c_{\mathrm{max}})$ the smallest nontrivial meager ideal?
- A stronger regularity condition on functions could perhaps produce a smaller meager ideal. But: analytic is too strong; differentiable is open!

$$
\begin{array}{ccccc}
 & & & & 2^{\aleph_0} \\
 & & & & \uparrow \\
 & & & & \mathfrak{hm}(c_{\max}) \\
 & & & & \uparrow \\
\mathrm{Cov}(\mathcal{L}ip(\mathbb{R})) \geq & \mathrm{Cov}(\mathcal{L}ip(\omega^\omega)) & = \mathrm{Cov}(\mathcal{L}ip(2^\omega)) & = \!=\!=\!= & \mathfrak{hm}(c_{\min}) \\
 & \uparrow & & & \\
\mathrm{Cov}(\mathcal{C}ont(\mathbb{R})) = & \mathrm{Cov}(\mathcal{C}ont(\omega^\omega)) & = \mathrm{Cov}(\mathcal{C}ont(2^\omega)) & & \\
 & \uparrow & & & \\
 & \mathfrak{d} & & &
\end{array}
$$