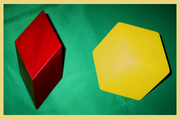
DONALD

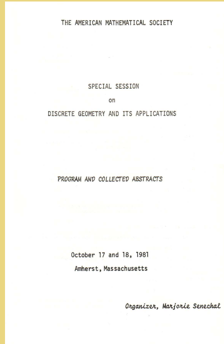
AND THE

GOLDEN RHOMBOHEDRA



Marjorie Senechal
Smith College
Northampton, MA

Models: RHOMBO, a geometric "toy" designed by Michael Longuet-Higgins . . . and much enjoyed by Donald



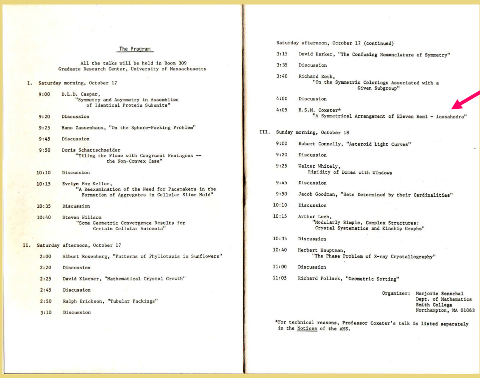
THE AMERICAN MATHEMATICAL SOCIETY

SPECIAL SESSION
on
DISCRETE GEOMETRY AND ITS APPLICATIONS

PROGRAM AND COLLECTED ABSTRACTS

October 17 and 18, 1981
Amherst, Massachusetts

Organizer, Marjorie Senechal



The Program

All the talks will be held in Room 309
Graduate Research Center, University of Massachusetts

I. Saturday morning, October 17

9:00 D.L.D. Caspar,
"Symmetry and Asymmetry in Assemblies of Identical Protein Subunits"

9:20 Discussion

9:25 Hans Zassenhaus, "On the Sphere-Packing Problem"

9:45 Discussion

9:50 Doris Schattschneider
"Tiling the Plane with Congruent Pentagons -- the Non-Convex Case"

10:10 Discussion

10:15 Evelyn Fox Keller,
"A Reexamination of the Need for Pacemakers in the Formation of Aggregates in Cellular Slime Mold"

10:35 Discussion

10:40 Steven Willson
"Some Geometric Convergence Results for Certain Cellular Automata"

II. Saturday afternoon, October 17

2:00 Alburt Rosenberg, "Patterns of Phyllotaxis in Sunflowers"

2:20 Discussion

2:25 David Klarner, "Mathematical Crystal Growth"

2:45 Discussion

2:50 Ralph Erickson, "Tubular Packings"

3:10 Discussion

Saturday afternoon, October 17 (continued)

3:15 David Harker, "The Confusing Nomenclature of Symmetry"

3:35 Discussion

3:40 Richard Roth,
"On the Symmetric Colorings Associated with a Given Subgroup"

4:00 Discussion

4:05 H.S.M. Coxeter*
"A Symmetrical Arrangement of Eleven Hemi - icosahedra"

III. Sunday morning, October 18

9:00 Robert Connelly, "Asteroid Light Curves"

9:20 Discussion

9:25 Walter Whitely,
Rigidity of Domes with Windows

9:45 Discussion

9:50 Jacob Goodman, "Sets Determined by their Cardinalities"

10:10 Discussion

10:15 Arthur Loeb,
"Modularly Simple, Complex Structures: Crystal Systematics and Kinship Graphs"

10:35 Discussion

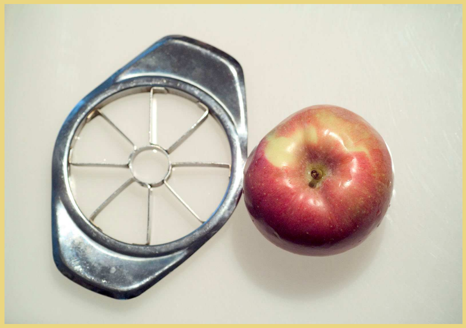
10:40 Herbert Hauptman,
"The Phase Problem of X-ray Crystallography"

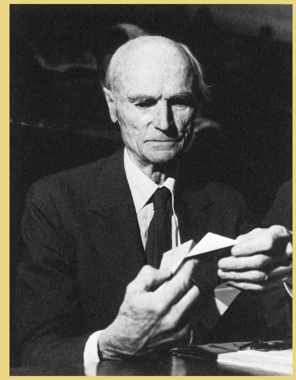
11:00 Discussion

11:05 Richard Pollack, "Geometric Sorting"

Organizer: Marjorie Senechal
Dept. of Mathematics
Smith College
Northampton, MA 01063

*For technical reasons, Professor Coxeter's talk is listed separately in the Notices of the AMS.



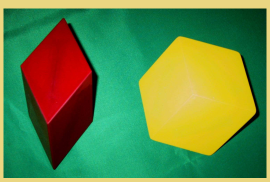


Smith College

April, 1984

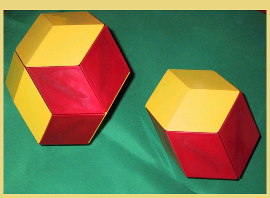
The Five Golden Isozonohedra

A6
acute rhombus

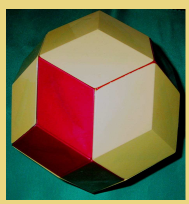


O6
obtuse rhombus

F20
Federov rhombic icosahedron



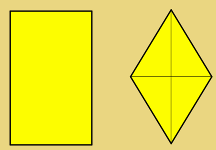
Bilinski rhombic dodecahedron **B12**



K30 Kepler triacontahedron

A rectangle having sides in the ratio τ:1 is a *golden rectangle*, a rhombus having diagonals in this ratio is a *golden rhombus*.

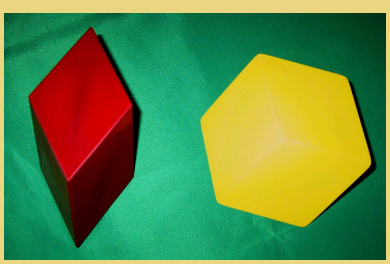
HSM Coxeter, "The rhombic triacontahedron," *Symmetry 2000*, edited by I. Hargittai and T. C. Laurent, Portland Press Ltd., 2002.



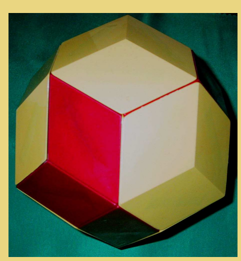
$\tau = (1 + \sqrt{5})/2$

Golden rectangle   Golden rhomb

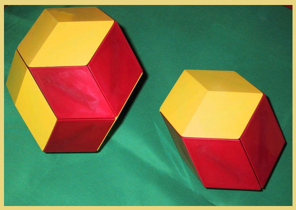
And a rhombohedron whose faces are six congruent golden rhombi is a ***golden rhombohedron*.** Such a rhombohedron has an axis of trigonal symmetry and is said to be obtuse or acute according to the nature of the three face angles at a vertex on this axis. *Coxeter, 2002, contd.*

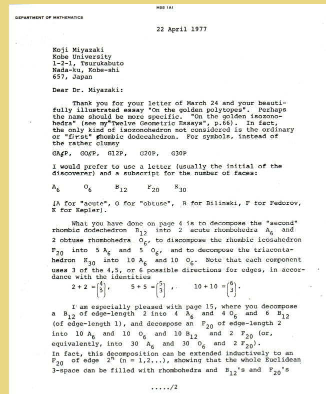


The ***triacontahedron***'s face has diagonals in the "golden section" ratio 1:τ. A model can be built up from twenty rhombohedra, ten acute and ten obtuse, bounded by such rhombs. *Regular Polytopes*, 2nd edition, 1963



By removing one of the p zones from a rhombic zonohedron and bringing together the two remaining pieces of the surface, one obtains a simpler zonohedron, with p-1 replacing p. In this manner the triacontrahedron yields the ***rhombic icosahedron*** . . . This in turn has a zone of eight faces, which can be removed so as to yield the two halves of a ***rhombic dodecahedron*** (p = 4) -- not Keplers ...





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DEPARTMENT OF MATHEMATICS

22 April 1977

Koji Miyazaki
Kobe University
1-2-1, Tsurukabuto
Nada-ku, Kobe-shi
657, Japan

Dear Dr. Miyazaki:

Thank you for your letter of March 24 and your beautifully illustrated essay "On the golden polytopes". Perhaps the name should be more specific. "On the golden isozonohedra" (see my "Twelve Geometric Essays", p.66). In fact, the only kind of isozonohedron not considered is the ordinary or "first" rhombic dodecahedron. For symbols, instead of the rather clumsy

GA6P, GO6P, G12P, G20P, G30P

I would prefer to use a letter (usually the initial of the discoverer) and a subscript for the number of faces:

A_6 O_6 B_{12} F_{20} K_{30}

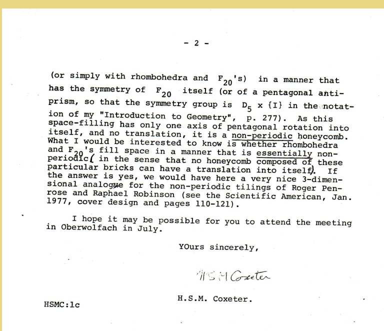
(A for "acute", O for "obtuse", B for Bilinski, F for Fedorov, K for Kepler).

What you have done on page 4 is to decompose the "second" rhombic dodechedron B_{12} into 2 acute rhombohedra A_6 and 2 obtuse rhombohedra O_6, to discompose the rhombic icosahedron F_{20} into 5 A_6 and 5 O_6, and to decompose the triacontahedron K_{30} into 10 A_6 and 10 O_6. Note that each component uses 3 of the 4,5, or 6 possible directions for edges, in accordance with the identities

$$2 + 2 = \binom{4}{3}, \quad 5 + 5 = \binom{5}{3}, \quad 10 + 10 = \binom{6}{3}.$$

I am especially pleased with page 15, where you decompose a B_{12} of edge-length 2 into 4 A_6 and 4 O_6 and 6 B_{12} (of edge-length 1), and decompose an F_{20} of edge-length 2 into 10 A_6 and 10 O_6 and 10 B_{12} and 2 F_{20} (or, equivalently, into 30 A_6 and 30 O_6 and 2 F_{20}). In fact, this decomposition can be extended inductively to an F_{20} of edge 2^n (n = 1,2...), showing that the whole Euclidean 3-space can be filled with rhombohedra and B_{12}'s and F_{20}'s

...../2



- 2 -

(or simply with rhombohedra and F_{20}'s) in a manner that has the symmetry of F_{20} itself (or of a pentagonal antiprism, so that the symmetry group is $D_5 \times \{I\}$ in the notation of my "Introduction to Geometry", p. 277). As this space-filling has only one axis of pentagonal rotation into itself, and no translation, it is a non-periodic honeycomb. What I would be interested to know is whether rhombohedra and F_{20}'s fill space in a manner that is essentially non-periodic (in the sense that no honeycomb composed of these particular bricks can have a translation into itself). If the answer is yes, we would have here a very nice 3-dimensional analogue for the non-periodic tilings of Roger Penrose and Raphael Robinson (see the Scientific American, Jan. 1977, cover design and pages 110-121).

I hope it may be possible for you to attend the meeting in Oberwolfach in July.

YOurs sincerely,

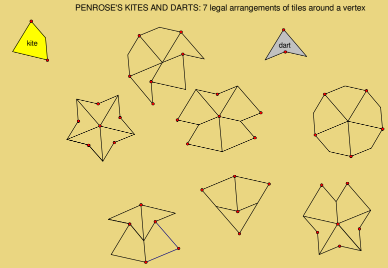
H.S.M. Coxeter

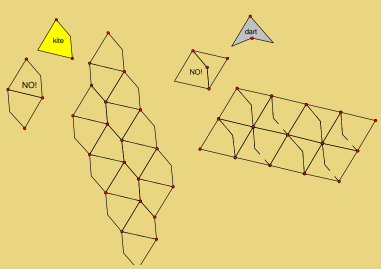
H.S.M. Coxeter.

HSMC:lc

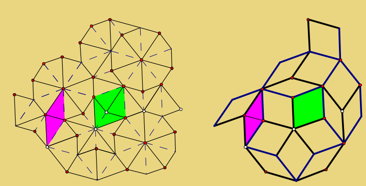
Donald was unaware of Robert Ammann's 3-D Penrose analogues.

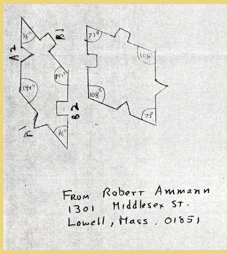
PENROSE'S KITES AND DARTS: 7 legal arrangements of tiles around a vertex

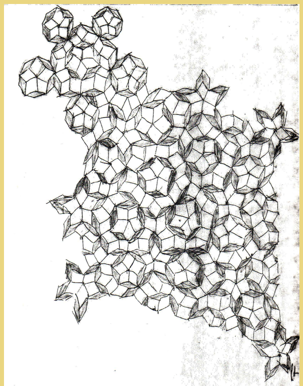




From kites and darts to thick and thin rhombs (and vice versa)







Robert Ammann to Martin Gardner, April 1976

Penrose tilings can be generated in (at least) four equivalent ways:

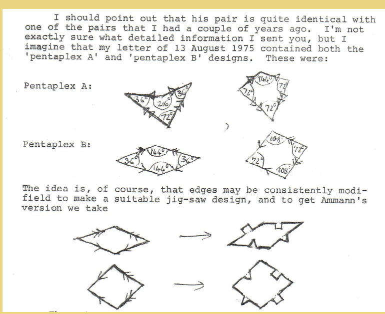
Fitting tiles together as prescribed by matching rules

Decomposition/inflation or composition

Duals of Ammann bar grids or de Bruijn pentagrids

Projection from the hypercube tiling of E^5

All tilings built with any of these methods are **locally isomorphic**, i.e, any patch in any tiling is relatively dense in that tiling and in all the others.



I should point out that his pair is quite identical with one of the pairs that I had a couple of years ago. I'm not exactly sure what detailed information I sent you, but I imagine that my letter of 13 August 1975 contained both the 'pentaplex A' and 'pentaplex B' designs. These were:

Pentaplex A:

Pentaplex B:

The idea is, of course, that edges may be consistently modified to make a suitable jig-saw design, and to get Ammann's version we take

Roger Penrose to Martin Gardner, Xxxx 1976

184 *Quasicrystals and geometry*

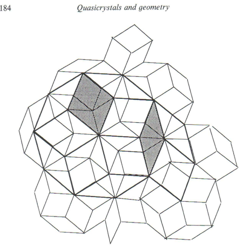


Fig. 6.13 Composition for the Penrose rhombs.

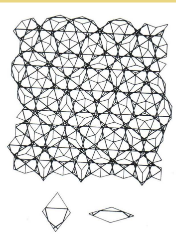
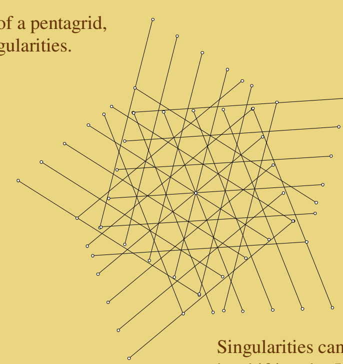
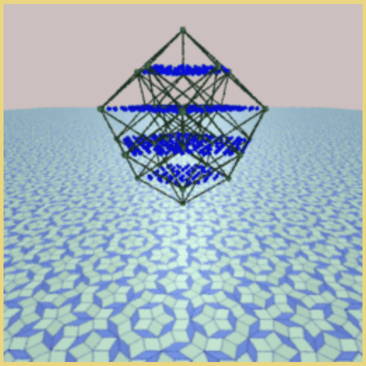


FIG. 1.22 Illustration of the dual operation that relates a 2-dim Penrose lattice to an Ammann grid.

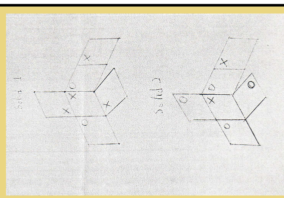
Portion of a pentagrid, with singularities.



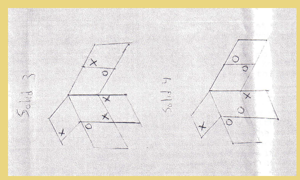
Singularities can be avoided by shifting the 5 grids



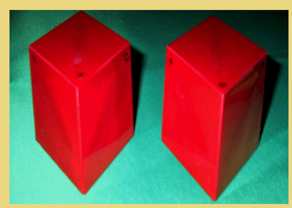
Robert Ammann had also discovered the first set of (apparent) 3-D aperiodic tiles. The tile shapes were the golden rhombohedra, A6 and 06, each marked into two (mirror-image) ways. In his first (spring 1976) letter to Martin Gardner, Ammann included marked nets the four tiles (X for bumps, O for dents).

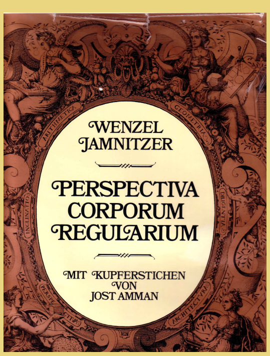


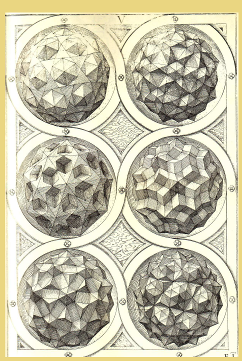
Robert Ammann to Martin Gardner, April 1976



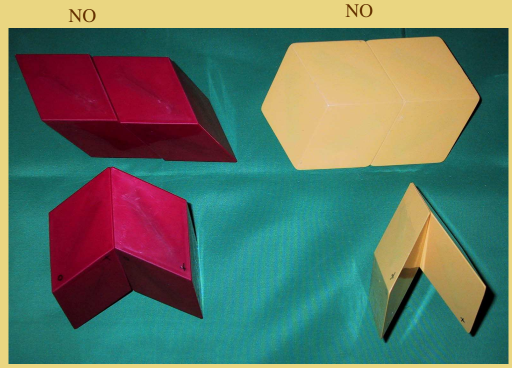
Robert Ammann to Martin Gardner, April 1976



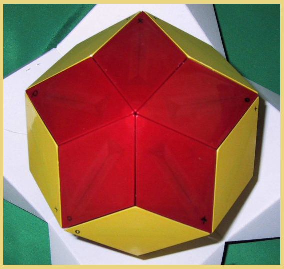
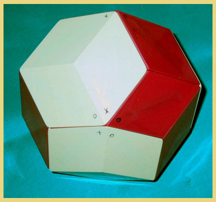




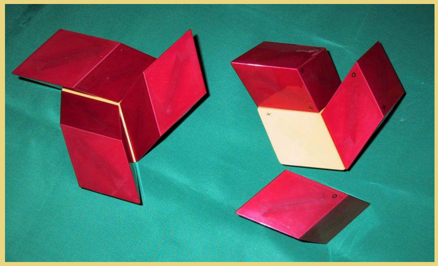
NO NO



YES YES

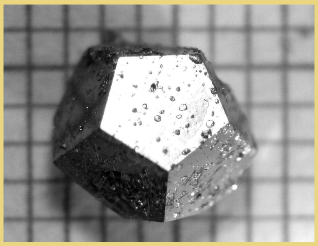



Ammann's matching rules permit
the five golden isozonohedra . . .



. . . . .but not the "other" triacontahedron K*

Fisher Research Group
Geballe Laboratory for Advanced Materials
Dept. of Applied Physics
Stanford University
CA 94305-4045



Al-Pd-Re single quasicrystal shown over a mm scale.

The flurry of tiling research following the discovery of quasicrystals showed that, in contrast to Penrose tilings in the plane, the various generating methods:

fitting tiles together as prescribed by matching rules

decomposition/inflation or composition

duals of Ammann plane grids or de Bruijn hexagrids

projection from the hypercube tiling of E^6

are, in general, **not** equivalent in three dimensions.

For example:

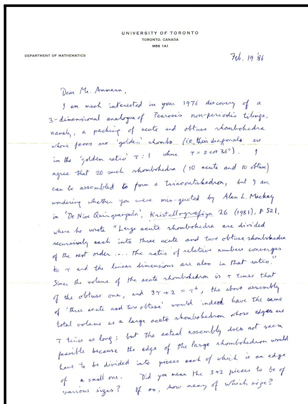
Katz showed that if the tiling is generated by the projection method, the two rhombohedra must be marked in twenty two different ways.

Socolar and Steinhardt showed that the golden isozonohedra are duals of "singular points" in "quasiperiodic hexagrids" and the singularities cannot be eliminated by shifting the grids.

Planes meeting at a point	dual configuration
six	triacontahedron
five	icosahedron
four	dodecahedron
three	acute rhombohedron

Socolar proved that Ammann's matching rules **do** force nonperiodicity but -- it ***seems*** -- they are not as powerful as the rules for the 2-D Penrose tiles.

For example, it ***may*** be possible to use them to construct tilings that are not locally isomorphic.

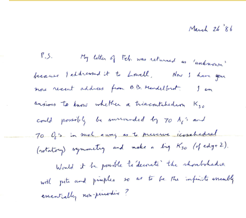

UNIVERSITY OF TORONTO
TORONTO, CANADA
M5S 1A1

DEPARTMENT OF MATHEMATICS

Feb. 19 '86

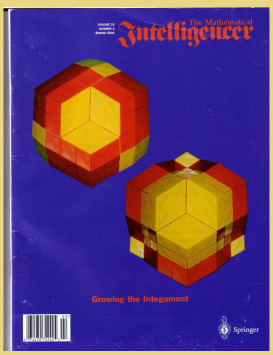
Dear Mr. Ammann,

I am much interested in your 1976 discovery of a 3-dimensional analogue of Penrose's non-periodic tilings, namely, a packing of acute and obtuse rhombohedra whose faces are 'golden' rhombs (i.e. their diagonals are in the 'golden ratio' $\tau : 1$ where $\tau = 2\cos 36°$). I agree that 20 such rhombohedra (10 acute and 10 obtuse) can be assembled to form a triacontahedron, but I am wondering whether you were mis-quoted by Alan L. Mackay in 'De Nive Quinquangula', Kristallografiya 26 (1981), p. 521, where he wrote "Large acute rhombohedra are divided recursively each into three acute and two obtuse rhombohedra of the next order .... the ratio of relative numbers converges to τ and the linear dimensions are also in that ratio." Since the volume of the acute rhombohedron is τ times that of the obtuse one, and $3\tau + 2 = \tau^4$, the above assembly of 'three acute and two obtuse' would indeed have the same total volume as a large acute rhombohedron whose edges are τ times as long; but the actual assembly does not seem feasible because the edge of the large rhombohedron would have to be divided into pieces each of which is an edge of a small one. Did you mean the 3+2 pieces to be of various sizes? If so, how many of which size?

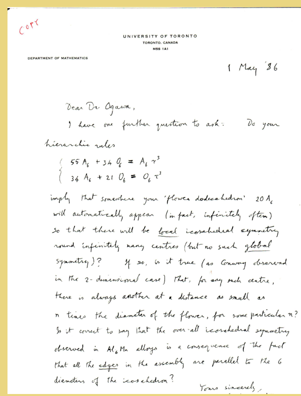

March 24 '86

P.S. My letter of Feb. was returned as 'unknown' because I addressed it to Lowell. Now I have your more recent address from B.B. Mandelbrot. I am anxious to know whether a triacontahedron K_{30} could possibly be surrounded by 70 A_6's and 70 O_6's in such a way as to preserve icosahedral (rotatory) symmetry and make a big K_{30} (of edge 2).

Would it be possible to 'decorate' the rhombohedra with pits and pimples so as to be the infinite assembly essentially non-periodic?



Michael Longuet-Higgins' construction of increasingly larger triacontahedra does not obey Ammann's rules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UNIVERSITY OF TORONTO
TORONTO, CANADA
M5S 1A1

DEPARTMENT OF MATHEMATICS

1 May '86

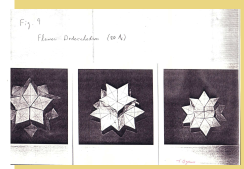
Dear Dr Ogawa,

I have one further question to ask: Do your hierarchic rules

$$\begin{cases} 55 A_6 + 34 O_6 = A_6 \tau^3 \\ 34 A_6 + 21 O_6 = O_6 \tau^3 \end{cases}$$

imply that somewhere your 'flower dodecahedron' 20 A_6 will automatically appear (in fact, infinitely often) so that there will be local icosahedral symmetry round infinitely many centres (but no such global symmetry)? If so, is it true (as Conway observed in the 2-dimensional case) that, for any such centre, there is always another at a distance as small as n times the diameter of the flower, for some particular n? Is it correct to say that the over-all icosahedral symmetry observed in Al_6Mn alloys is a consequence of the fact that all the edges in the assembly are parallel to the 6 diameters of the icosahedron?

Yours sincerely,



A proposal to the Geometry Center,1996

Three-dimensional discrete geometry lies at the heart of many fundamental problems in mathematics and other sciences. For example, the rapidly growing field of polytope theory is important in many branches of discrete mathematics. For polytopes, there are intriguing problems that connect geometric with combinatorial or algebraic properties. . . .

The discovery of quasicrystals in physics in 1984 spurred vigorous research activity in the mathematics of long-range aperiodic order. **Many aperiodic 3D structures are so complex that exploration of their local and global properties is infeasible with the visualization tools currently available.**

The complexity of geometric structures that arise in scientific and industrial applications requires better understanding. The need to build, visualize, manipulate, and deform these 3D structures in an interactive computer environment is widely recognized.

The major areas that would directly benefit from such 3D software include convexity, tilings, quasicrystals and aperiodicity, lattices and periodicity, packing and covering, groups and symmetry, polyhedra and incidence structures, oriented matroids, reflection groups and hyperplane arrangements, illumination problems, molecular structures, frameworks and rigidity (ball and stick structures), and growth systems (foams, cell growth).

Existing software either enables the researcher to visualize known structures and to give analytic (coordinate) input to build some new structures (Geomview and Mathematica) or provides graphical tools that allow one to construct objects free-hand (CAD/CAM). No existing integrated package provides the interactivity and mathematical content needed to study complex geometric problems in 3D.

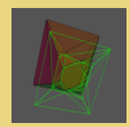
We propose to develop a 3D package analogous to two-dimensional programs such as The Geometer's Sketchpad, which allow the researcher to build 2D geometric figures in a ``hands-on'' manner, and to explore the implications of geometric constraints.

We want to construct 3D polytopes from scratch by graphically fitting geometric pieces together and to be able to interactively manipulate and view these objects at every stage of their construction. This involves being able to perform important operations such as selecting vertices, edges, faces, and polytopes, grouping and ungrouping these objects, and coloring them and moving them around independently or as a group. ***We need to be able to intersect and dissect the basic building blocks and attach them to each other at angles to generate larger figures in space such as 3D geometric complexes, tilings, or clusters, with the ability to also reverse the construction process and dissect figures.*** In building up figures from smaller units, we need to be able to specify constraints (such as perpendicularity, {\em parallelism}, dihedral angles, or flexible attachment) as we build.

The planning group:

Heidi Burgiel, The Geometry Center
Daniel Huson, Universität Bielefeld, Germany
Nicholas Jackiw, Key Curriculum Press, Inc., Berkeley
Stuart Levy, The Geometry Center
Jesus A. de Loera, Geometry Center
Robert Moody, University of Alberta
Jiri Patera, Universite de Montreal
Michelle Raymond, The Geometry Center
Doris Schattschneider, Moravian College
Egon Schulte, Northeastern University
Marjorie Senechal, Smith College,

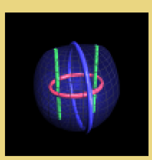
Date: Mon, 22 Apr 96
From: senechal
To: schulte@neu.edu



Dear Egon,

I'll call Dick McGehee soon, but before that it would be good to settle on the project name. Naming things makes them real somehow and a good name might go a long way to persuading people to bring it into existence. Can you take votes on that? Several people emailed that they like the name Coxeter Project (or Project Coxeter) but not everyone has done so. If the group does decide on Coxeter, would it be appropriate for me to call him? Or may be better for you to do it? I think you know him better than I do and so might be more persuasive.

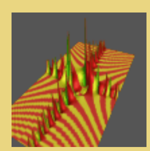
Date: Mon, 22 Apr 1996
From: Robert V. Moody
To: discgeom@geom.UMN.edu
Subject: Coxeter
Dear All:



The Coxeter Project sounds good. It would be important to get his blessing for the name. There is a possible acronym from his initials: Hands-on Synthetic Manipulation (of 3d graphics). I am pretty sure he would not approve of that!

Bob

Date: Mon, 22 Apr 1996
From: Marjorie Senechal
To: Robert V. Moody
Cc: discgeom@geom.UMN.edu
Subject: Re: HSM



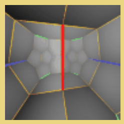
Hi Bob,

Coxeter might not approve of his initials being interpreted as "Hands-on Synthetic Manipulation" but I'll be his wife would! I've heard her tell him several times that he should get with it and learn to use computers. She could talk him into giving it his blessing even if the HSM part is just a joke.

Date: Wed, 24 Apr 1996
From: Doris Schattschneider
To: discgeom@geom.UMN.edu
Subject: Re: Coxeter's approval

Thanks, Egon-- this is great news. Perhaps an earlier (but not well-known) precedent is Escher's use of Coxeter's name as a verb. When Escher was working on drawings for his Circle Limit hyperbolic prints he wrote to his son George that he was "Coxetering."

Doris



Date: Fri, 26 Apr 96
From: senechal
To: discgeom@umn.edu
Subject: conversation with Dick McGehee

I just had a long talk with Dick McGehee. He is very interested in Project Coxeter and encourages us to submit a proposal to the Geometry Center. From his discussions with industrial and other scientists he knows that the kind of 3D interaction we hope to develop is urgently needed in a wide variety of problems and applications -- not only in polytope theory! At the same time, the fact that Project Coxeter will address needs of the discrete geometry research community is crucial, since the Center was established to address the visualization needs of researchers. . . .

Date: 6 Jun 1996 08:01:38 -0700
From: Nick Jackiw
To: Marjorie Senechal
Subject: Project Coxeter & Sketchpad

Next season we plan to focus on 3-D. In a recent discussion with Dick McGehee, I learned that the Center will likely go ahead with Project Coxeter, but that (as you forwarned) they can't address any educational component or research in the pitch they make to the NSF. Dick was very interested in the possibility of a Sketchpad / Coxeter relationship to accelerate and exploit Coxeter's educational opportunities. . . . .

Date: Wed, 19 Jun 1996
From: SCHULTE
To: discgeom@geom.UMN.edu
Subject: our project
Dear Project Coxeter team:

We are up to a bumpy start! I assume that everyone received Marjorie's message about the newest development at the Geometry Center and its consequences for our project. After the good work at our meeting and the successful proposal writing afterwards, this was a somewhat unexpected and disappointing news. I cannot add much to what Marjorie said, but it seems that we cannot really do much about it at this point in time.

From the message I got the impression that the Center's main nervousness is the next site visit by an NSF team. Until then, or at least until it is clear that the Center can "produce results" for the next visit, we cannot expect too much in the way of support.

Eight years later . . . . .

Date: Tue, 27 Apr 2004 12:14:30 -0400
From: Nicholas Jackiw
To: Marjorie Senechal
Subject: Re: 3-D Sketchpad?

Alas, the 3D-GSP project went nowhere fast. . . . what we really need is new interaction technology, at the hardware level. Another avenue of pursuit was to determine if any of the (relatively inexpensive) 3-D manipulation hardware available from home gaming market manufacturers were accessible to educational software development. The quick answer from Nintendo and Sega: to do anything educational would instantly destroy our credibility with our 14-year-old male customer base.