J. Mather — "Near a Double Resonance"

$$L = L(\theta, \dot\theta, t) = \ell_0(\dot\theta) + \varepsilon P(\theta, \dot\theta, t)$$

$d^2 \ell_0 > 0 \qquad \|P\|_{C^r} = 1$

$\theta \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \qquad \dot\theta \in B^2_R =$ ball of radius R

$\beta_L : H_1(\mathbb{T}^2; \mathbb{R}) = \mathbb{R}^2 =$ "space of frequencies" $\to \mathbb{R}$

$\beta_L(\omega) = \min_\mu \{A(\mu) : \rho(\mu) = \omega\}$ "minimal average action"

μ Euler-Lagrange invariant probability measure

$A(\mu)$ average action $A(\mu) = \int L\, d\mu$

$\rho(\mu)$ rotation vector

Convex conjugate $\alpha_L : H^1(\mathbb{T}^2; \mathbb{R}) \to \mathbb{R}$

$$\alpha_L(c) = -\min_{\omega \in H_1} \{\beta_L(\omega) - \langle \omega, c\rangle\}.$$

Legendre - Fenchel transform

$\mathcal{L} = \mathcal{L}_{\beta_L} : H_1 \to \{$compact, convex, non-empty subsets of $H^1\}$.

$\mathcal{L}(\omega) = \{c : \alpha_L(c) + \beta_L(\omega) = \langle \omega, c\rangle\}$.

Note: $\forall c, \quad \alpha_L(c) + \beta(\omega) \geq \langle \omega, c\rangle$.

$\Gamma \subset \mathbb{R}^2$ line segment

resonance condition $\vec{k} = (k_0, k_1, k_2) \in \mathbb{Z}^3$
$(k_1, k_2) \neq (0, 0)$.

$\omega \in \Gamma \Rightarrow k_0 + k_1\omega_1 + k_2\omega_2 = 0.$

$\mathbb{T}^2_{\vec{k}} = \{(\theta_1, \theta_2, t) \in \mathbb{T}^2 \times \mathbb{T} : k_1\theta_1 + k_2\theta_2 + k_0 t \equiv 0 \text{ mod. } 1\}$

$\mathbb{T}^1_{\vec{k}} = \mathbb{T}^2 \times \mathbb{T} / \mathbb{T}^2_{\vec{k}}.$

$P_{\omega, \vec{k}}(\varphi) = \langle P(\theta, \omega, t) \rangle_{(\theta, t) \in \varphi}.$

$\varphi =$ coset of $\mathbb{T}^2_{\vec{k}}$.

First genericity conditions:

- $\forall \omega \in \Gamma$, $P_\omega(\varphi)$ has at most two global minima m_ω, m_ω'.
- global minima non-degenerate (Morse)
- If two m_ω, m_ω', $\frac{d}{d\omega} P_\omega(m_\omega) \neq \frac{d}{d\omega} P_\omega(m_\omega')$.
(transversality condition)

Note: For all but finitely many $\omega \in \Gamma$, only one global minimum.

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double resonance condition: $\exists \vec{l} = (l_0, l_1, l_2) \in \mathbb{Z}^3$
$(l_1, l_2) \neq (0, 0)$ $\vec{k}, \vec{l}$ linearly independent

$l_0 + l_1\omega_1 + l_2\omega_2 = 0$

$\{$doubly resonant $\omega \in \Gamma\}$ countable dense subset of Γ.
$\Gamma \cap \overset{\shortparallel}{\mathbb{Q}^2}$

ω° doubly resonant

$$\mathbb{T}^1_{\omega^\circ} = \{(\lambda\omega_1^\circ, \lambda\omega_2^\circ, \lambda) \in \mathbb{T}^2 \times \mathbb{T} : \lambda \in \mathbb{R}\}$$
$$= \{(\theta_1, \theta_2, t) \in \mathbb{T}^2 \times \mathbb{T} : k_1\theta_1 + k_2\theta_2 + k_0 t$$
$$= l_1\theta_1 + l_2\theta_2 + l_0 t \equiv 0 \text{ mod. } 1\}.$$

$\mathbb{T}^1_{\omega^\circ} \subset \mathbb{T}^2_{\vec{k}}$.

<u>Def'n</u> $\mathbb{T}^2_{\omega^\circ} = \mathbb{T}^2 \times \mathbb{T} / \mathbb{T}^1_{\omega^\circ}$.

$P_{\omega^\circ}(\mathcal{C}) = \langle P(\theta, \omega, t) \rangle_{(\theta, \omega) \in \mathcal{C}}$ $\quad$ $\mathcal{C}$ coset of $\mathbb{T}^1_{\omega^\circ}$.

$P_{\omega^\circ} : \mathbb{T}^2_{\omega^\circ} \longrightarrow \mathbb{R}$

$\mathbb{T}^2_{\vec{k}} / \mathbb{T}^1_{\omega^\circ} \subset \mathbb{T}^2_{\omega^\circ}$ $\quad$ 1-torus

$h_0 = [\mathbb{T}^2_{\vec{k}} / \mathbb{T}^1_{\omega^\circ}] \in H_1(\mathbb{T}^2_{\omega^\circ}; \mathbb{Z}) = H_1(\mathbb{T}^2_{\omega^\circ}; \mathbb{R})$.

$\pi : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2_{\omega^\circ}$ projection

$(\theta, \tau) \in \mathbb{T}^2 \times \mathbb{T}$ $\quad$ $d\pi_{(\theta, \tau)} : \mathbb{R}^2 = T_{(\theta, \tau)}(\mathbb{T}^2 \times 0)$
$\to T_{\pi(\theta, \tau)} \mathbb{T}^2_{\omega^\circ}$ isomorphism

$g_{\omega^\circ} = d^2 l_0(\omega^\circ)$ = constant Riemannian metric on $\mathbb{T}^2_{\omega^\circ}$

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$K_{\omega^0} = g_{\omega^0}/2$

$L_{\omega^0} = K_{\omega^0} + P_{\omega^0}$ conservative mechanical system on $\mathbb{T}^2_{\omega^0}$.

action minimizing measures of L_{ω^0} with non-zero rotation vector in $\mathbb{R}\cdot h_0$ are supported on periodic orbits.

Maupertius principle

$$g_{\omega^0,E} = (E + P_{\omega^0})\, g_{\omega^0}, \qquad E \geq E_0 = -\min P_{\omega^0}$$

$g_{\omega^0,E}$ Riem metric on $\mathbb{T}^2_{\omega^0}$, $E > E_0$.

Carneiro: μ c-minimal for $L_{\omega^0} \Rightarrow$

$\alpha_{L_{\omega^0}}(c) = E =$ energy of any orbit in supp μ

minimal measure with rotation vector in $\mathbb{R}\cdot h_0$

$=$ shortest $g_{\omega^0,E}$ geodesic in h_0.

Futher genericity conditions (Strongly doubly resonant ω^0).

- $\forall E \geq E_0$, $g_{\omega^0,E}$ has at most two shortest geodesics γ_E, γ_E' in h_0.
- These are non-degenerate (Morse)
- $\frac{d}{dE}\,\text{length}(\gamma_E) \neq \frac{d}{dE}\,\text{length}(\gamma_E')$.

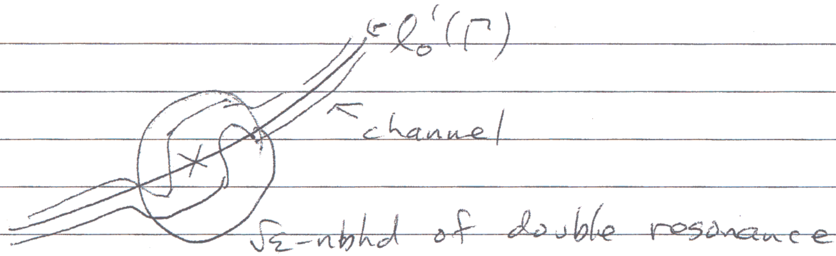
• P_ω has a unique minimum, non-degenerate (Morse)

cusp residual set defined by genericity conditions (only strong double resonances) and $0<\varepsilon\le\varepsilon_0$.

Proposition $\varepsilon P \in$ cusp-residual set $\Longrightarrow$

$\mathcal{L}_{\beta_L}(\Gamma)$ channel

$C_0\sqrt{\varepsilon} \le$ width of channel $\le C_1\sqrt{\varepsilon}$.



channel $\setminus$ $\cup$($\sqrt{\varepsilon}$-nbhds of all double resonances) $\supset l_0'(\Gamma)$.

$c \in$ interior of channel

$\Longrightarrow M_c$ (= proj. on $\mathbb{T}^2\times\mathbb{T}$ of support of c-minimal measures) $\subset$

$\subset$ small nbhd. of $\pi^{-1}(m_\omega)$, $\pi : \mathbb{T}^2\times\mathbb{T}\to\mathbb{T}^1_{\vec{k}}$,

or $\pi^{-1}(\gamma_E)$ $\pi : \mathbb{T}^2\times\mathbb{T}\to\mathbb{T}^2_{\omega^0}$.

Moreover, M_c is a Denjoy minimal set or periodic orbit

Denjoy minimal set supports a unique invariant measure $\Longrightarrow$ variationally connected.