NONDESTRUCTIVE ESTIMATION OF QUANTUM RELATIVE ENTROPY

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Effective Density Matrix

- ρ – a density matrix defined on a Hilbert space $\mathcal{H}$;
- $\{|\lambda_i\rangle\}$ and $\{\lambda_i\}$ – the eigenbasis and the eigenvalues of ρ

Definition 1 *For every ρ and orthonormal basis $\{|a_i\rangle\}$, Jozsa and Presnell defined a so-called "effective" density matrix*

$$\tilde{\rho} \triangleq \sum_j \mu_j |a_j\rangle\langle a_j|,$$

with respect to the basis $\{|a_i\rangle\}$, where $\mu_j \triangleq \sum_k M_{jk}\lambda_k$ and $M_{jk} \triangleq \langle a_j|\lambda_k\rangle\langle\lambda_k|a_j\rangle \geqslant 0$.

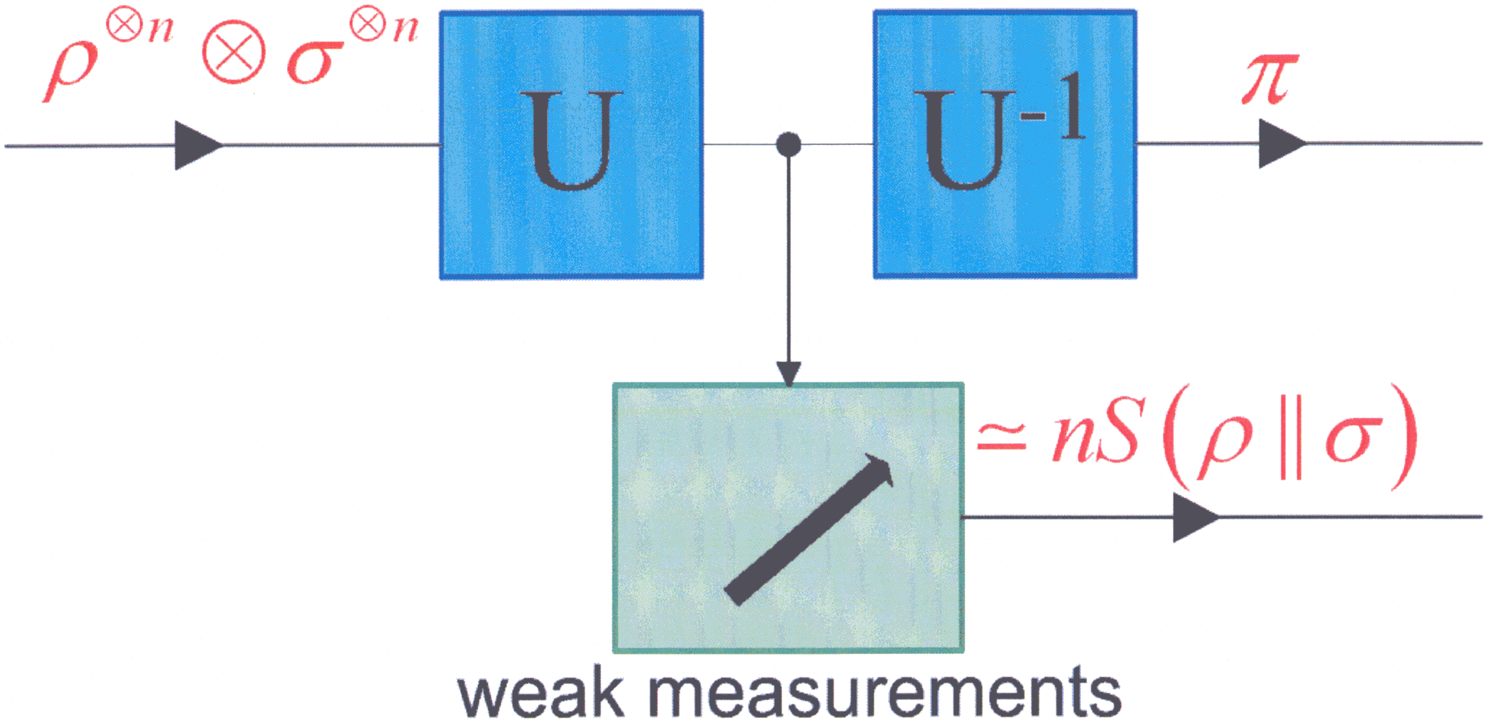
Turning Classical Code to Quantum

- $\{|a_i\rangle\} \triangleq \{|a_1\rangle, |a_2\rangle, \ldots, |a_{|\mathcal{A}|}\rangle\}$ – an arbitrary, but fixed orthonormal basis of the Hilbert space $\mathcal{H}$;
- $\{|a_i\rangle\}_{in}^{\otimes n}$ and $\{|a_i\rangle\}_{out}^{\otimes n}$ – orthonormal bases in "input" Hilbert space $\mathcal{H}^{\otimes n}$ and "output" Hilbert space $\mathcal{H}_{out}^{\otimes n}$, respectively.

For any classical code φ, we define an unitary operator $U_\varphi^n : \mathcal{H}^{\otimes n} \to \mathcal{H}_{out}^{\otimes n}$ by the bases vectors mapping

$$U_\varphi^n|x^n\rangle = |\varphi(x^n)\rangle,$$

where $|x^n\rangle \in \{|a_i\rangle\}_{in}^{\otimes n}$ and $|\varphi(x^n)\rangle \in \{|a_i\rangle\}_{out}^{\otimes n}$ for all $x^n \in \mathcal{A}^n$.



$\rho^{\otimes n} \otimes \sigma^{\otimes n}$ → U → U^{-1} → π

$\simeq nS(\rho \| \sigma)$

$\pi \to \rho^{\otimes n} \otimes \sigma^{\otimes n}$ as $n \to \infty$

ρ and σ are unknown!

Quantum relative entropy:

$$S(\rho \| \sigma) \triangleq Tr\rho \log \rho - Tr\rho \log \sigma$$

Relations between Quantum and Classical Entropies

Suppose we have two quantum i.i.d. sources with density matrices ρ and σ.

Lemma 1

$$S(\rho||\sigma) = D(p_{\tilde{\rho}}||p_\sigma) + H(p_{\tilde{\rho}}) - S(\rho) = D(p_{\tilde{\rho}}||p_\sigma) + H(p_{\tilde{\rho}}) - H(p_\rho),$$

where $\{|\chi_i\rangle\}$ *is the eigenbasis of* σ
$\tilde{\rho}$ *is the effective density matrix of* ρ *with respect to* $\{|\chi_i\rangle\}$ *and*
p_ρ *stands for the probability distribution defined by eigenvalues of* ρ

$$H(q) \triangleq -\sum_{x \in \mathcal{A}} q(x) \log q(x) \qquad D(q||p) \triangleq \sum_{x \in \mathcal{A}} q(x) \log \frac{q(x)}{p(x)}$$

Univ. Estimation of Quantum Relative Entropy

$$S(\rho||\sigma) = D(p_{\tilde{\rho}}||p_{\sigma}) + H(p_{\tilde{\rho}}) - H(p_{\rho})$$

$$S(\rho||\sigma) = \left\langle \frac{1}{n} |\mathbf{C}_{H+D}(z^n, x^n)| \right\rangle_{p_{\tilde{\rho}\otimes\sigma}} - \left\langle \frac{1}{n} |\mathbf{C}_{H}(x^n)| \right\rangle_{p_{\rho}}$$

Classical Estimation of $H(q)$ and $D(q||p)$

Ziv and Merhav introduced a pair of universal estimation codes $\mathbf{C}_H(\cdot)$ and $\mathbf{C}_{H+D}(\cdot,\cdot)$. For any $\delta > 0$ and all sufficiently large n

$$q_z\left(z^n : \left|\frac{1}{n}|\mathbf{C}_H(z^n)| - H(q)\right| > \delta\right) \leqslant \exp[-n\ cf(\delta)];$$

$$u_{zx}\left((z^n, x^n) : \left|\frac{1}{n}|\mathbf{C}_{H+D}(z^n, x^n)| - H(q) - D(q||p)\right| > \delta\right) \leqslant \exp[-n\ cf(\delta)],$$

where c is a constant; $u_{zx}(z^n, x^n) \triangleq p_x(x^n)q_z(z^n)$ is a joint probability measure; $f(\cdot)$ at zero neighborhood is continuous, non-decreasing, non-negative function of known order; $f(0) = 0$.

Weak Measurements

Theorem 1 *Suppose a product state* $|x^n\rangle \in \mathcal{H}_{in}$*, emitted by a quantum i.i.d. source with density matrix* ρ*, is subjected to a unitary transformation* U_φ^n *with a computational basis* $\{|a_i\rangle\}$.

Let $\tilde{\rho}$ *be the effective matrix of* ρ *with respect to* $\{|a_i\rangle\}$*. Let* φ *be a one-to-one mapping. If, for any sufficiently small* $\delta > 0$*, the inequality below is satisfied, then, for all sufficiently large* n*, by weak measurements of* $U_\varphi^n|x^n\rangle$*, one can estimate the value of* $\left\langle \frac{1}{n} |\varphi(x^n)| \right\rangle_{p_{\tilde{\rho}}}$ *with arbitrary high accuracy while maintaining the fidelity arbitrary close to the unity, where* $\langle \cdots \rangle_{p_{\tilde{\rho}}}$ *denotes the average w.r.t. the measure* $p_{\tilde{\rho}}$.

$$p_{\tilde{\rho}}\left(\left| \frac{1}{n} |\varphi(x^n)| - \left\langle \frac{1}{n} |\varphi(x^n)| \right\rangle_{p_{\tilde{\rho}}} \right| > \delta \right) < \exp[-n\ cf(\delta)]$$

The Algorithm

$$S(\rho||\sigma) = \left\langle \frac{1}{n} |\mathbf{C}_{H+D}(z^n, x^n)| \right\rangle_{p_{\tilde{\rho}\otimes\sigma}} - \left\langle \frac{1}{n} |\mathbf{C}_H(x^n)| \right\rangle_{p_\rho}$$

Suppose we have two quantum i.i.d. sources with a priory unknown density matrices ρ and σ, where $\{|\lambda_i\rangle\}$ is the eigenbasis of ρ and $\{|\chi_i\rangle\}$ is the eigenbasis of σ. Let $\tilde{\rho}$ be the effective density matrix of ρ with respect to $\{|\chi_i\rangle\}$.

1. By weak measurements estimate $\{|\lambda_i\rangle\}$ and $\{|\chi_i\rangle\}$
2. for classical code $\mathbf{C}_H$, define $U^n_{\mathbf{C}_H} := U^n_\varphi\big|_{\varphi=\mathbf{C}_H}$ with computational basis $\{|\lambda_i\rangle\}$, then get an estimate of $\left\langle \frac{1}{n} |\mathbf{C}_H(x^n)| \right\rangle_{p_\rho}$ by weak measurements.
3. for classical code $\mathbf{C}_{H+D}$, define $U^n_{\mathbf{C}_{H+D}} := U^n_\varphi\big|_{\varphi=\mathbf{C}_{H+D}}$ with computational basis $\{|\chi_i\rangle\} \otimes \{|\chi_i\rangle\}$, then get an estimate of $\left\langle \frac{1}{n} |\mathbf{C}_{H+D}(z^n, x^n)| \right\rangle_{p_{\tilde{\rho}\otimes\sigma}}$ by weak measurements.

As n goes to infinity, the algorithm will converge to $S(\rho||\sigma)$ by continuity of classical entropy $H(\cdot)$ and classical relative entropy $D(\cdot||\cdot)$