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Good Additive Cyclic Quantum Codes

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Abstract - The paper presents all the best additive cyclic quantum codes of length up to 23 qubits, as well as a table showing the existed additive cyclic quantum codes of length up to 31 qubits.

1. Introduction

Calderbank's paper [1] turned the problem of finding additive quantum codes to the problem of finding self-orthogonal cods over $GF(4)^n$. In [1], many methods were presented. In this paper, we use one of those methods in [1] to do a thorough research.

2. Some theorems

Calderbank's paper [1] presented the following theorem about the additive cyclic codes over $GF(4)^n$:

Theorem 1:

a) Any $(n, 2^k)$ additive cyclic code C has two generators which can be represented as $\langle wp(x)+q(x), r(x)\rangle$, where $p(x), q(x), r(x)$ are binary polynomials, $p(x)$ and $r(x)$ divide x^n-1 (mod 2), $r(x)$ divides $q(x)(x^n-1)/p(x)$ (mod 2), and $k = 2n - \deg p - \deg r$.

b) If $\langle wp'(x)+q'(x), r'(x)\rangle$ is another such representation, then $p'(x) = p(x), r'(x) = r(x)$ and $q'(x) \equiv q(x)$ (mod $r(x)$).

c) C is self-orthogonal if and only if

$$p(x)r\left(x^{n-1}\right) \equiv p\left(x^{n-1}\right)r(x) \equiv 0 \quad (\text{mod } x^n-1)$$

$$p(x)q\left(x^{n-1}\right) \equiv p\left(x^{n-1}\right)q(x) \quad (\text{mod } x^n-1)$$

This theorem enables us to search all of the self-orthogonal additive cyclic codes over

$GF(4)^n$. In order to find the corresponding $[[n, n-k, d]]$ additive cyclic quantum codes, we need the following theorem, which is also mentioned in [1]:

Theorem 2: If C is an $(n.2^k)$ additive code with weight enumerator $W_C(x, y)$ [2], then the weight enumerator of $C^{\perp}$ is given by:

$$W_{C^{\perp}}(x, y) = 2^{-k} W(x+3y, x-y)$$

We can find the minimum distance of $C^{\perp} - C$ by comparing the coefficients of $W_C(x, y)$ with those of $W_{C^{\perp}}(x, y)$.

The search ranges for the polynomials $p(x), q(x), r(x)$ are the following:

1) The arrange for $p(x)$ is between 1 and $x^n - 1$, not including $x^n - 1$. $p(x)$ can not be 0, for if $p(x)$ is 0, the code C will be a binary code.

2) The arrange for $r(x)$ is between 1 and $x^n - 1$, including $x^n - 1$. When $r(x)$ is $x^n - 1$, $r(x)$ can not be considered as a generator, for $r(x)$ is actually 0 (mod $x^n - 1$). In this case, the generator of the code is simply $\langle wp(x) + q(x) \rangle$.

3) The arrange for $q(x)$ is between 1 and $r(x)$, including $r(x)$. Note that $q(x) = r(x)$ is equivalent to $q(x) = 0$, for the generators $\langle wp(x) + r(x), r(x) \rangle$ and $\langle wp(x), r(x) \rangle$ generate same code.

3. The search algorithm

1) Find all of the irreducible binary factors of $x^n - 1$ over $GF(2)$. These factors will help us in the next step – finding $p(x)$ and $r(x)$.

2) Consider all of the pairs of $p(x)$ and $r(x)$ which satisfy the equation

$$p(x)r(x^{n-1}) \equiv p(x^{n-1})r(x) \equiv 0 \pmod{x^n - 1}$$

3) For each pair of $p(x)$ and $r(x)$ coming from step 2), consider all of the possible $q(x)$ which satisfy

   a) $q(x)(x^n - 1) \equiv 0 \pmod{p(x)r(x)}$

   b) $p(x)q(x^{n-1}) \equiv p(x^{n-1})q(x) \pmod{x^n - 1}$

4) For each set of qualified polynomials $p(x), q(x), r(x)$, we calculate the weight enumerators of the code and its dual code to find d.

4. The results

Table 1.1

Additive cyclic quantum codes with highest minimum distance

Parameters	Generators
$[[5,0,3]]$	$\langle w\ w\ 0\ 1\ 0\rangle\ \langle 1\ 1\ 1\ 1\ 1\rangle$
$[[5,1,3]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 1\rangle$
$[[5,4,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\rangle$
$[[7,0,3]]$	$\langle w\ w\ w\ 0\ w\ 0\ 0\rangle\ \langle 1\ 0\ 1\ 1\ 0\ 0\ 0\rangle$
$[[7,1,3]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 0\ 1\rangle$
$[[7,3,2]]$	$\langle \bar{w}\ w\ 0\ \bar{w}\ 0\ 1\ 1\rangle$
$[[7,4,2]]$	$\langle \bar{w}\ 1\ w\ w\ \bar{w}\ 0\ 1\rangle$
$[[7,6,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\rangle$
$[[9,0,4]]$	$\langle \bar{w}\ \bar{w}\ w\ 1\ 0\ 1\ 0\ 0\ 0\rangle\ \langle 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\rangle$
$[[9,1,3]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 0\ 0\ 0\ 1\rangle$
$[[9,2,3]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ 1\ 0\ 1\ 1\ 0\ 1\rangle$

$[[9,3,3]]$	$\langle \bar{w}\ 0\ 0\ \bar{w}\ 1\ 1\ 0\ 1\ 1 \rangle$
$[[9,6,2]]$	$\langle \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1 \rangle$
$[[9,7,1]]$	$\langle \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0 \rangle$
$[[9,8,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[11,0,4]]$	$\langle \bar{w}\ \bar{w}\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0 \rangle$ $\langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[11,1,3]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \rangle$
$[[11,10,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[13,0,5]]$	$\langle w\ w\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0 \rangle$ $\langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[13,1,5]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1 \rangle$
$[[13,12,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[15,0,6]]$	$\langle w\ 1\ w\ \bar{w}\ w\ 1\ w\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$ $\langle 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \rangle$
$[[15,1,5]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \rangle$
$[[15,2,5]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1 \rangle$
$[[15,3,5]]$	$\langle \bar{w}\ 0\ 0\ \bar{w}\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0 \rangle$
$[[15,4,4]]$	$\langle \bar{w}\ w\ 1\ 0\ w\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1 \rangle$
$[[15,5,4]]$	$\langle \bar{w}\ 0\ w\ 1\ w\ \bar{w}\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1 \rangle$
$[[15,6,4]]$	$\langle \bar{w}\ 0\ 0\ \bar{w}\ \bar{w}\ w\ \bar{w}\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1 \rangle$
$[[15,7,3]]$	$\langle \bar{w}\ \bar{w}\ 0\ w\ 0\ 0\ 1\ w\ 0\ 1\ 0\ 1\ 1\ 1\ 1 \rangle$
$[[15,8,3]]$	$\langle \bar{w}\ 1\ 1\ 1\ w\ 0\ \bar{w}\ \bar{w}\ w\ 0\ 0\ 1\ 1\ 0\ 1 \rangle$
$[[15,9,3]]$	$\langle \bar{w}\ w\ 0\ 0\ \bar{w}\ w\ w\ 1\ 1\ w\ 1\ 0\ 1\ 1\ 1 \rangle$

$[[15,10,2]]$	$\langle \bar{w}\ 1\ 1\ 1\ 1\ \bar{w}\ 1\ 1\ 1\ 1\ \bar{w}\ 1\ 1\ 1\ 1 \rangle$
$[[15,11,2]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 1\ \bar{w}\ \bar{w}\ 1\ 0\ 1\ \bar{w}\ \bar{w}\ 1\ 0\ 1 \rangle$
$[[15,12,2]]$	$\langle \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1 \rangle$
$[[15,13,1]]$	$\langle \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0 \rangle$
$[[15,14,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[17,0,7]]$	$\langle w\ w\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0 \rangle \quad \langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[17,1,7]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1 \rangle$
$[[17,8,4]]$	$\langle \bar{w}\ 1\ 0\ \bar{w}\ w\ \bar{w}\ 0\ 1\ \bar{w}\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1 \rangle$
$[[17,9,4]]$	$\langle \bar{w}\ \bar{w}\ 1\ w\ 1\ 1\ w\ 1\ \bar{w}\ \bar{w}\ 0\ 0\ 1\ 0\ 1\ 0\ 0 \rangle$
$[[17,16,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[19,0,7]]$	$\langle w\ w\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \rangle \quad \langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[19,1,7]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \rangle$
$[[19,18,1]]$	$\langle \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w} \rangle$
$[[21,0,8]]$	$\langle w\ w\ 0\ 1\ 0\ \bar{w}\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle, \langle 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \rangle$
$[[21,1,7]]$	$\langle \bar{w}\ \bar{w}\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \rangle$
$[[21,2,6]]$	$\langle \bar{w}\ w\ \bar{w}\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1 \rangle$
$[[21,3,6]]$	$\langle \bar{w}\ w\ 0\ w\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1 \rangle$
$[[21,4,6]]$	$\langle \bar{w}\ 0\ w\ \bar{w}\ w\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \rangle$
$[[21,5,6]]$	$\langle \bar{w}\ 1\ 1\ 0\ w\ \bar{w}\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \rangle$
$[[21,6,5]]$	$\langle \bar{w}\ w\ w\ 0\ \bar{w}\ 1\ \bar{w}\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1 \rangle$

$[[21,7,5]]$	$\langle \bar{w}\ 0\ 0\ w\ w\ \bar{w}\ \bar{w}\ \bar{w}\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \rangle$
$[[21,8,4]]$	$\langle \bar{w}\ 0\ w\ 1\ 0\ \bar{w}\ 0\ w\ w\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1 \rangle$
$[[21,9,4]]$	$\langle \bar{w}\ 0\ 0\ 0\ 1\ 1\ w\ 1\ 1\ w\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1 \rangle$
$[[21,10,4]]$	$\langle \bar{w}\ 0\ w\ 1\ \bar{w}\ 0\ w\ w\ 0\ 1\ w\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1 \rangle$
$[[21,11,4]]$	$\langle \bar{w}\ 0\ 1\ w\ w\ 0\ 1\ 1\ 1\ w\ 0\ w\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \rangle$
$[[21,12,3]]$	$\langle \bar{w}\ 1\ w\ \bar{w}\ 1\ 1\ w\ 0\ \bar{w}\ 0\ 0\ w\ w\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1 \rangle$
$[[21,13,3]]$	$\langle \bar{w}\ \bar{w}\ w\ 0\ w\ 1\ \bar{w}\ \bar{w}\ \bar{w}\ \bar{w}\ 0\ w\ 0\ w\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \rangle$
$[[21,14,3]]$	$\langle \bar{w}\ w\ 0\ 0\ 0\ \bar{w}\ w\ w\ 1\ w\ \bar{w}\ w\ 1\ 0\ w\ 0\ 1\ 0\ 1\ 1\ 1 \rangle$
$[[21,15,3]]$	$\langle \bar{w}\ 1\ \bar{w}\ 1\ 1\ w\ 0\ 1\ \bar{w}\ \bar{w}\ 0\ 0\ w\ 1\ w\ w\ 0\ 1\ 1\ 0\ 1 \rangle$
$[[21,16,2]]$	$\langle \bar{w}\ \bar{w}\ w\ \bar{w}\ w\ 1\ w\ 0\ \bar{w}\ 1\ 0\ w\ w\ 1\ 0\ 0\ w\ 0\ 1\ 1\ 1 \rangle$
$[[21,17,2]]$	$\langle \bar{w}\ w\ 0\ \bar{w}\ 0\ 1\ 1\ \bar{w}\ w\ 0\ \bar{w}\ 0\ 1\ 1\ \bar{w}\ w\ 0\ \bar{w}\ 0\ 1\ 1 \rangle$
$[[21,18,2]]$	$\langle \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1\ \bar{w}\ 1\ 1 \rangle$
$[[21,19,1]]$	$\langle \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0\ \bar{w}\ \bar{w}\ 0 \rangle$
$[[21,20,1]]$	$\langle \bar{w}\ \bar{w} \rangle$
$[[23,0,8]]$	$\langle w\ w\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \rangle$, $\langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[23,1,7]]$	$\langle w\ w\ w\ w\ w\ 0\ 0\ w\ 0\ 0\ w\ 0\ w\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$, $\langle 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$
$[[23,11,4]]$	$\langle w\ w\ \bar{w}\ w\ w\ 0\ 0\ \bar{w}\ 0\ 1\ \bar{w}\ 0\ \bar{w}\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$, $\langle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$
$[[23,12,4]]$	$\langle w\ \bar{w}\ w\ w\ \bar{w}\ 1\ 0\ \bar{w}\ 0\ 1\ w\ 0\ \bar{w}\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$
$[[23,22,1]]$	$\langle \bar{w}\ \bar{w} \rangle$

Table 1.2

All of the valid $[[25 \le n \le 31, k]]$ for additive cyclic quantum codes

("E" means exist)

$k \backslash n$	25	27	29	31
0	E	E	E	E
1	E	E	E	E
2		E		
3		E		
4	E			
5	E			E
6		E		E
7		E		
8		E		
9		E		
10				E
11				E
12				
13				
14				
15				E
16				E
17				
18		E		
19		E		
20	E	E		E
21	E	E		E
22				
23				
24	E	E		
25		E		E
26		E		E
27				
28			E	
29				
30				E

Table 1.3
Some additive cyclic code with highest minimum distance for $n = 31$

Parameters	Generators
$[[31,15,5]]$	$\langle \bar{w}\ w\ \bar{w}\ w\ 0\ 0\ \bar{w}\ 1\ w\ \bar{w}\ 1\ w\ 0\ \bar{w}\ 1\ 1\ w\ 0\ 0\ \bar{w}\ 1\ w\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$, $\langle 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \rangle$
$[[31,16,5]]$	$\langle w\ \bar{w}\ \bar{w}\ \bar{w}\ 1\ 0\ w\ 1\ w\ \bar{w}\ 1\ w\ 1\ w\ 1\ 0\ w\ 1\ 0\ w\ 1\ w\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rangle$, $\langle 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \rangle$
$[[31,20,4]]$	$\langle \bar{w}\ 1\ 0\ 0\ 1\ w\ \bar{w}\ 0\ 1\ \bar{w}\ 0\ \bar{w}\ w\ 0\ 1\ \bar{w}\ w\ \bar{w}\ 0\ \bar{w}\ w\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 \rangle$
$[[31,21,4]]$	$\langle \bar{w}\ w\ 0\ 0\ 1\ \bar{w}\ 0\ \bar{w}\ 1\ w\ w\ w\ 1\ \bar{w}\ 1\ w\ 0\ 1\ w\ \bar{w}\ 1\ \bar{w}\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \rangle$

5. Conclusion

All of the codes listed in table 1.1, except the codes $[[11,0,4]]$, $[[11,1,3]]$, meet the lower bounds in the table of [1], which are the best additive quantum codes we can achieve now. Thus, to a great degree, searching the best quantum codes can be replaced by searching the best additive cyclic quantum codes. The search complexity will therefore be greatly reduced.

[1] A. R. Calderbank, E. M. Rains, P. W. Shor and N. J. A. Sloane, "Quantum Error Correction Via Codes Over GF(4)," *IEEE Trans. on Info. Theory*, vol. 44, no. 4, pp. 1369-1387, July 1998.

[2]. F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes* (NorthHolland, Amsterdam, 1977).