Feedback Control of Continuous Projective Measurement

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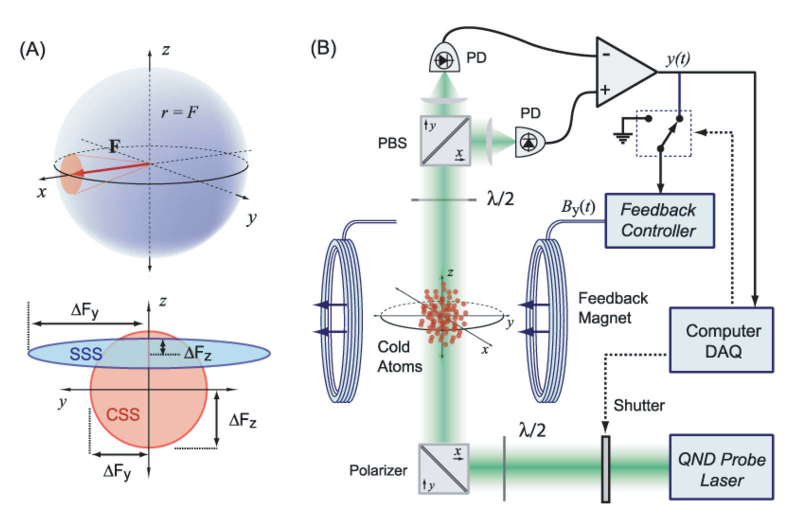
Introduction

It has long been recognized that measurement can be used as a non-deterministic means of preparing quantum states that are otherwise difficult to obtain. With discrete projective measurements, one must typically accept a probabilistic outcome. However, with certain continuous QND models of projective measurement, the observer can affect the result by using feedback control. To illustrate this concept, we here present experimental results demonstrating deterministic preparation of spin squeezed states via measurement and control. We then consider the theoretical extension of the conditioning equations at long times and propose feedback controllers capable of deterministically preparing highly entangled multi-particle Dicke states.

Experiment *Spin Squeezing at Short Times*

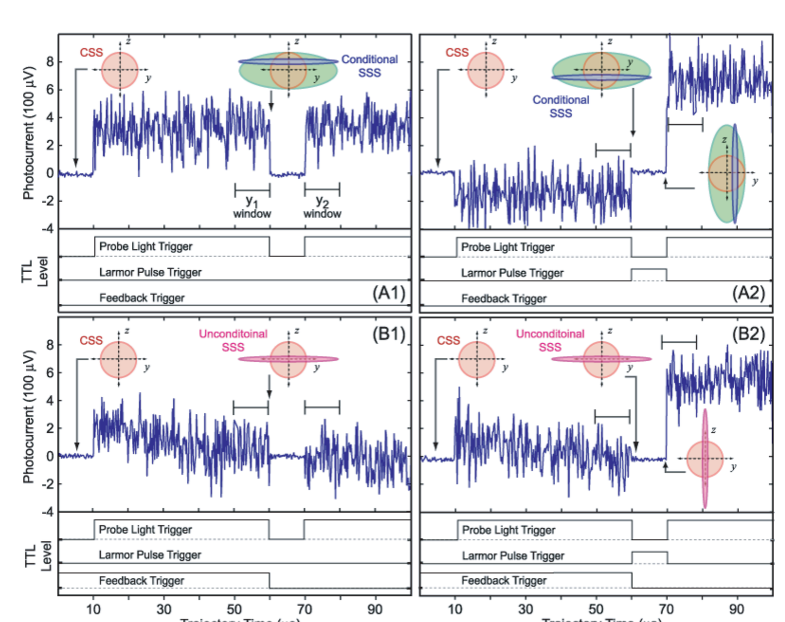
Schematic

An ensemble of Cesium atoms is laser cooled then optically pumped such that each atom occupies the F=4, m_F=4 ground state in the x basis. The initial collective state is thus a coherent spin state (CSS) as shown in (A). Subsequent to the state initialization, a far off resonant, linearly polarized probe beam traverses the sample. The polarization rotation is measured by a polarimeter and the resulting photocurrent provides continuous information about the collective J_z of the ensemble. As information is acquired, the variance in the z direction is deterministically reduced while the mean becomes randomly displaced. When control is enabled, the measurement is used to modulate the y magnetic field to cancel the mean projection.



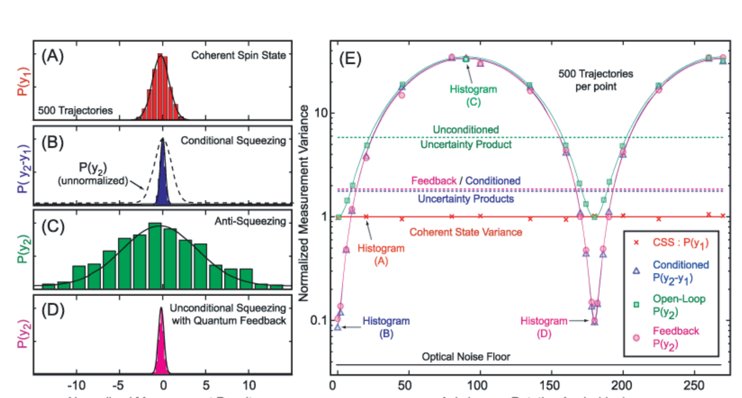
Measurement Trajectories

Example trajectories with corresponding timing diagram. If a constant field were present, large scale Larmor precession would be observed. The upper two plots demonstrate measurement projection noise, with an optional inter-field rotation to observe the anti-squeezing from the initial measurement. The lower two plots demonstrate the deterministic preparation of the spin-squeezing with the use of feedback control. Independent measurements have been made to verify the expected scaling of the projection noise with atom number.



Measurement Statistics

The measurements of the above plot were repeated 500 times at each of several inter-measurement rotation angles with and without feedback. The open-loop conditional measurements and the closed-loop measurements (using the above averaging windows) display the expected sinusoidal squeezing curve. Notice that the optical noise floor is below the squeezing minima.



Directions for further work

Future work will focus on optimizing the free space squeezing demonstrated here and identifying limiting noise sources. We have recently investigated quantum parameter estimation applications with this system, measuring an unknown magnetic field via Larmor precession at the same time the spin squeezing is produced. In future experiments we plan on applying a field parallel to the spins such that the system can be used to simulate non-QND oscillator-like dynamics and test related measurement procedures. Eventually, a cavity and an optical lattice will be added to the system to suppress spontaneous emission and extend the time during which the projective behavior of the measurement is valid.

Theory *Dicke State Preparation at LongTimes*

The free space measurement description eventually becomes complicated as spontaneous emission causes destruction of the quantum state. If this effect is suppressed, as it can be with a high finesse cavity, the QND projective behavior will eventually prepare an eigenstate of the measured variable, J_z. Here we analyze the long time behavior of the related conditioning equations and demonstrate that the projective measurement can in principle be made deterministic with feedback control.

Measurement Dynamics

Stochastic Master Equation (SME)

In homodyne configuration, the photocurrent is given by $y(t)dt = \langle J_z\rangle(t)dt + dW(t)/2\sqrt{M\eta}$

The stochastic master equation describing how to map this photocurrent to the optimal state description is given by

$$d\rho(t) = -i[H(t),\rho(t)]dt + \mathcal{D}[\sqrt{M}J_z]\rho(t)dt + \sqrt{\eta}\mathcal{H}[\sqrt{M}J_z]\rho(t)\left(2\sqrt{M\eta}[y(t)dt - \langle J_z\rangle dt]\right)$$

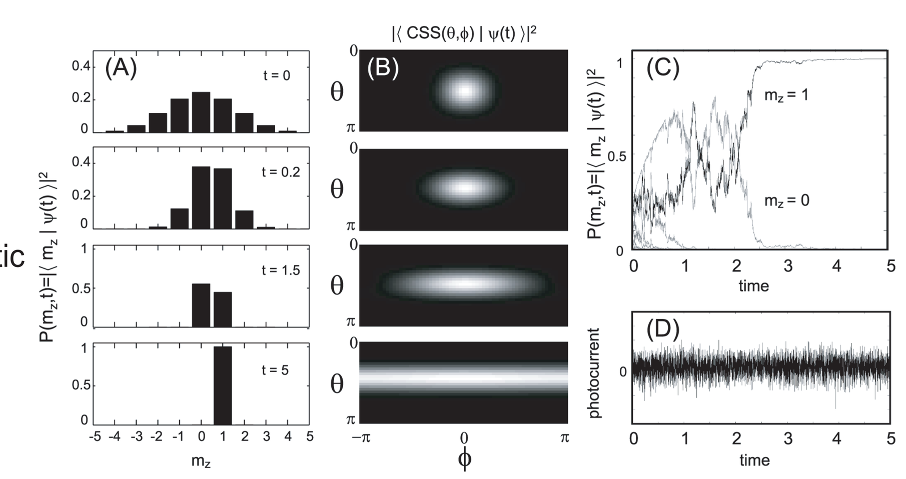
where

$$\begin{aligned}\mathcal{D}[c]\rho &\equiv c\rho c^\dagger - (c^\dagger c\rho + \rho c^\dagger c)/2\\ \mathcal{H}[c]\rho &\equiv c\rho + \rho c^\dagger - \mathrm{Tr}[(c+c^\dagger)\rho]\rho\end{aligned}$$

$\eta =$ quantum efficiency
$M =$ measurement rate

If the efficiency is one, we can use the stochastic Schrodinger equation (SSE) instead

$$\begin{aligned}d|\psi(t)\rangle &= (-iH(t) - M(J_z - \langle J_z\rangle(t))^2/2)|\psi(t)\rangle dt\\ &\quad + \sqrt{M}(J_z - \langle J_z\rangle(t))|\psi(t)\rangle dW(t)\end{aligned}$$



In this figure, an initially x-polarized coherent spin state (CSS) of N=10 spin-1/2 particles is evolved via the SSE. At small times, a conditional spin squeezed state is prepared. At long times, the state converges randomly to one of the fixed points of the SSE, which are the eigenstates of J_z (a.k.a. Dicke states) and can be highly entangled.

Dicke states

$$\begin{aligned}J_z|m\rangle &= m|m\rangle\\ \mathbf{J}^2|m\rangle &= N/2(N/2+1)|m\rangle\\ m &\in \{-N/2,...,N/2\}\end{aligned}$$

Unentangled Dicke states (CSS)

$$|m=+N/2\rangle = |\uparrow_1\uparrow_2\cdots\uparrow_N\rangle$$
$$|m=-N/2\rangle = |\downarrow_1\downarrow_2\cdots\downarrow_N\rangle$$

Highly entangled Dicke state

$$|m=0\rangle = C\,\Sigma_P P(|\uparrow_1\cdots\uparrow_{N/2}\downarrow_{N/2+1}\cdots\downarrow_N\rangle)$$

Moment Evolution

Now we consider the evolution of the moments of J_z which can be extracted from the above SSE/SME. First, we can show

$$d\langle J_z^n\rangle = 2\sqrt{M\eta}(\langle J_z^{n+1}\rangle - \langle J_z^n\rangle\langle J_z\rangle)dW(t)$$

which implies the martingale property

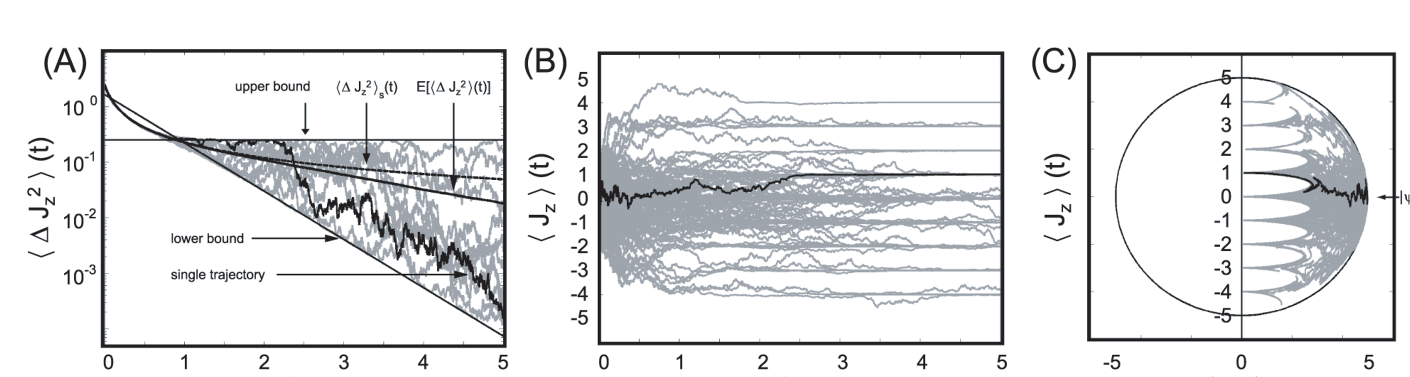
$$\mathrm{E}_s[\langle J_z^n\rangle(t)] = \langle J_z^n\rangle(s), \quad s \le t$$

It can also be shown that the variance decreases to zero on average, proving the eventual preparation of Dicke states by the open-loop SME.

$$\mathrm{E}[\langle\Delta J_z^2\rangle(t)] = \frac{\langle\Delta J_z^2\rangle(0)}{1+4M\eta\langle\Delta J_z^2\rangle(0)(t+\xi(t))}, \quad \xi(t) \ge 0$$

Through numerical simulation, we notice other interesting features of the dynamics. In (A) it is seen that the variance is bounded, such that the projection will always take a finite amount of time. Also, (C) clearly shows that certain regions of Hilbert space are forbidden by the dynamics of the SSE.

The long time variance bounds imply two level system behavior $\exp[-2(\eta Mt-1)]/4 < \langle\Delta J_z^2\rangle(t) \le 1/4$



Reduced representations

Short time moment expansion

$$\begin{aligned}d\langle J_z\rangle(t) &= \gamma\langle J_x\rangle(t)b(t)\,dt\\ &\quad + 2\sqrt{M\eta}\langle\Delta J_z^2\rangle(t)\,dW(t)\\ d\langle\Delta J_z^2\rangle(t) &= -4M\eta\langle\Delta J_z^2\rangle^2(t)\,dt\\ &\quad - i\gamma\langle[\Delta J_z^2, J_y]\rangle(t)b(t)\,dt\\ &\quad + 2\sqrt{M\eta}\langle\Delta J_z^3\rangle(t)dW(t)\end{aligned}$$

Current average estimator

$$\langle J_z\rangle_a(t) = \frac{\int_0^t y(s)ds}{t}$$

$$\begin{aligned}V_a &\equiv \mathrm{E}[(\langle J_z\rangle_a(t) - \langle J_z\rangle(t))^2] + \mathrm{E}[\langle\Delta J_z^2\rangle(t)]\\ &= \frac{1}{4M\eta t}\end{aligned}$$

Without a field, the discretization of the Dicke levels can be resolved by the sub-optimal current average estimator, but the simplicity of the estimator comes with a slower convergence rate.

Closed-Loop Dynamics

Now we wish to add feedback control to the measurement process in the hopes of preparing a given state deterministically on every trial. In the following we assume unity efficiency (SSE) and we aim to produce a particular desired Dicke state (m=m_d) for N=10. For the controller, we choose to work with Bayesian (state-based) feedback as it naturally turns the disturbing feedback off once the desired state is prepared, unlike Markovian (direct current) based feedback. We choose our feedback gain small enough such that the numerical results remain valid.

The 'cost function' that the control should minimize (for target state m_d) is given by the quantity:

$$\begin{aligned}U &\equiv (\langle J_z\rangle - m_d)^2 + \langle\Delta J_z^2\rangle\\ &= \Sigma_m\langle m|\rho|m\rangle^2(m-m_d)^2\\ &\ge 0\end{aligned}$$

Feedback Law 1

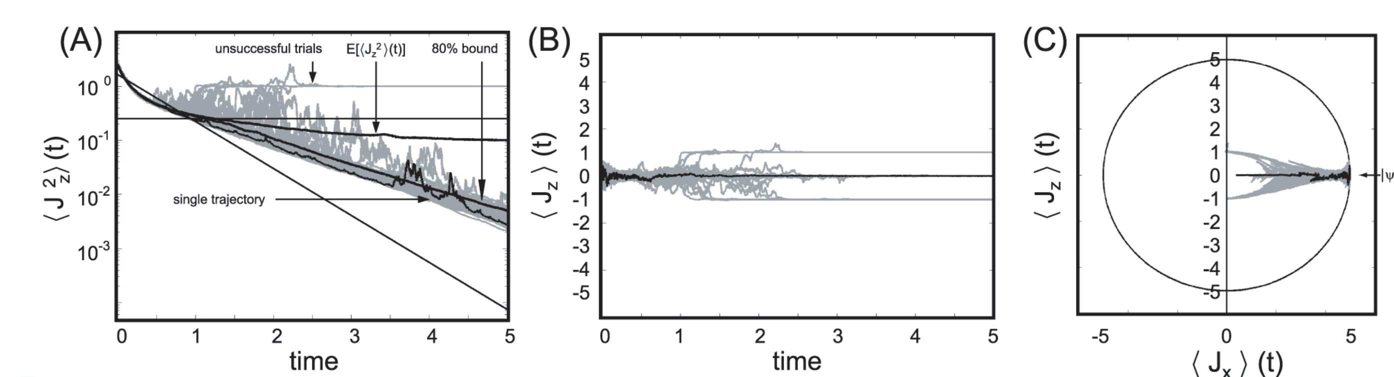
It can be shown that the average evolution of the cost function is given by

$$d\mathrm{E}[U(t)] = -2\gamma\mathrm{E}\left[b(t)\left(\frac{\langle J_xJ_z + J_zJ_x\rangle(t)}{2} - m_d\langle J_x\rangle(t)\right)\right]dt$$

This suggests that, to make this quantity negative, we choose a controller of the form

$$b_1(t) = \lambda\left(\frac{\langle J_xJ_z + J_zJ_x\rangle(t)}{2} - m_d\langle J_x\rangle(t)\right)$$

This controller (with a gain of 10) results in the evolution shown below with m_d=0. While the number of successful trials (m=0) is increased from 25 to 90 percent, the remaining fraction is lost because all Dicke states remain fixed points with this controller. Also notice that the presence of the control law somewhat compromises the 'best case' projection rate in (A).

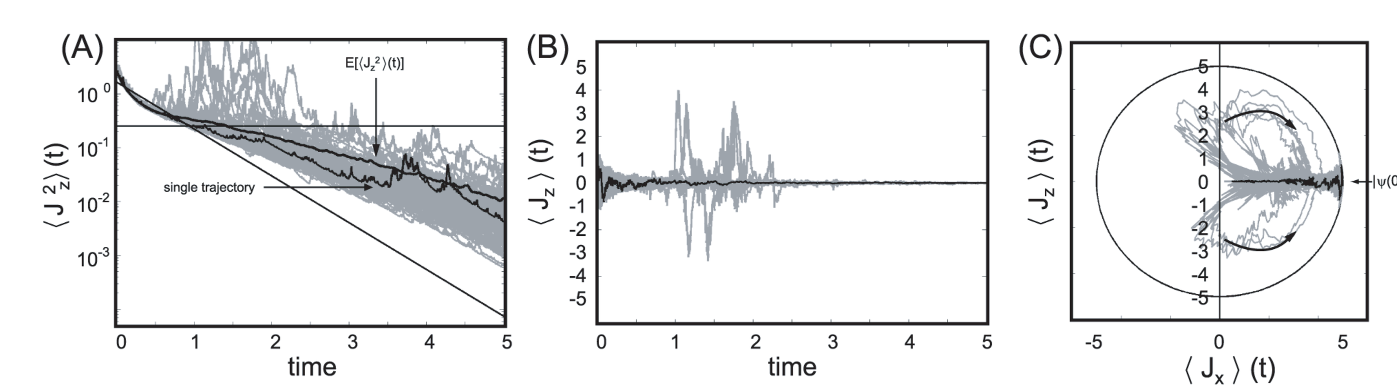


Feedback Law 2

Motivated by the previous results, we now choose a feedback controller that explicitly makes the desired state the *only* fixed point of the SSE/SME:

$$b_2(t) = \lambda(\langle J_z\rangle(t) - m_d)$$

While this controller is not explicitly designed to make the cost function decrease on average, we numerically demonstrate below that this law (again with a gain of 10) seems to prepare the same Dicke state (m_d=0) deterministically on every trial. This is shown by the exponentially decreasing average of the cost function in (A). Although some fraction (about 10 percent) of the trajectories 'miss' on the first pass, they are recycled by the control back into the attractive region of the target state.



Directions

The example presented here highlights several interesting lines of research in the field of state preparation via measurement and control:

Optimization - Given measurement dynamics, a cost function, an actuator, and experimental constraints (bandwidth, etc), we would like to find constructive methods for producing near optimal controller designs.
Model Reduction - A key element in the design of any controller is reducing its complexity such that it can be implemented by a device with finite resources.
Classical theory - Although quantum features often dominate, it is useful to adopt and adapt techniques already developed for classical estimation, control, stochastics, filtering, etc.
Experimental improvement - Only by attaining sufficiently quantum limited experiments can these ideas be tested.

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