MORE NOTES ABOUT THE DECOMPOSITION OF $Gl_n(\mathbb{C})$ IRREDUCIBLE MODULES INTO S_n IRREDUCIBLES

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First we list some properties of the Frobenius map which sends a an S_n character $\chi : S_n \to \mathbb{C}$ into a symmetric function by the map

$$\mathcal{F}(\chi) = \sum_{\lambda \vdash n} \chi(\sigma(\lambda)) p_\lambda / z_\lambda.$$

For an S_n module M, denote $Frob(M) = \mathcal{F}(char_{S_n}(M))$.

Let M^λ be a $Gl_n(\mathbb{C})$ module with character equal to the symmetric function $s_\lambda(x_1, x_2, \ldots, x_n)$.

I mentioned in the last writeup that $Frob(M^{(1^k)}) = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$. I received an elementary proof of this proposition from Adriano Garsia:

$$M^{(1^k)} \simeq \mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \simeq Ind_{S_k \times S_{n-k}}^{S_n} \mathcal{L}\{x_1 \wedge x_2 \wedge \cdots \wedge x_k\}$$

where the action of S_{n-k} is trivial on the module and S_k has the sign action on $x_1 \wedge x_2 \wedge \cdots \wedge x_k$.

Therefore $Frob(M^{(1^k)}) = s_{(1^k)} s_{(n-k)} = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$.

That two line proof should be broken down into lemmas

Lemma 1.

$$M^{(1^k)} = \mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

Proof. We compute the $Gl_n(\mathbb{C})$ character of $\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. Let $diag(y_1, y_2, \cdots, y_k)$ represent a diagonal matrix of $Gl_n(\mathbb{C})$ which acts on the variables by $diag(y_1, y_2, \cdots, y_k) x_i = y_i x_i$.

$$\sum_{i_1 < i_2 < \cdots < i_k} diag(y_1, y_2, \cdots, y_k) x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \Big|_{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}} =$$

$$\sum_{i_1 < i_2 < \cdots < i_k} y_{i_1} y_{i_2} \cdots y_{i_k} = s_{(1^k)}(y_1, y_2, \ldots, y_n)$$

Therefore $\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is the irreducible module with character $s_{(1^k)}(y_1, y_2, \ldots, y_n)$. □

Lemma 2.

$$\mathcal{L}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \simeq Ind_{S_k \times S_{n-k}}^{S_n} \mathcal{L}\{x_1 \wedge x_2 \wedge \cdots \wedge x_k\}$$

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Proof. Recall that $Ind_{S_k\times S_{n-k}}^{S_n}\mathcal{L}\{b_i\} = \mathcal{L}\{\sigma\otimes_{S_k\times S_{n-k}} b_i : \sigma\in S_n, b_i\}$ where we have the tensor *over* a group satisfies the relations

$$\sigma h\otimes_H v = \sigma\otimes_H hv.$$

The isomorphism is given by

$$x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_k}\mapsto \begin{pmatrix}1 & \cdots & k & k+1 & \cdots & n\\ i_1 & \cdots & i_k & j_1 & \cdots & j_{n-k}\end{pmatrix}\otimes_{S_k\times S_{n-k}} x_1\wedge x_2\wedge\cdots\wedge x_k$$

where $\{j_1, j_2, \ldots, j_{n-k}\} = [n]\backslash\{i_1, i_2, \ldots, i_k\}$.

The proof is to show that this map is equivariant with respect to the action of the symmetric group S_n and it suffices to show that the action of the simple transpositions $(j, j+1)$ are equal on both basis elements. There will be four cases to consider, namely both j and $j+1$ are in $\{i_1, i_2, \ldots, i_k\}$, j is in $\{i_1, i_2, \ldots, i_k\}$ and $j+1$ is not, $j+1$ is in $\{i_1, i_2, \ldots, i_k\}$ and j is not, and $j, j+1\notin\{i_1, i_2, \ldots, i_k\}$.

We leave the remainder of the proof as an exercise to the reader. □

The third step of Adriano's proof is that we need to know some properties of the Frobenius map. We list below some of the images of common S_n modules.

trivial S_n module	$\rightarrow$	$s_{(n)}$
sign S_n module	$\rightarrow$	$s_{(1^n)}$
permutation representation $\{1, 2, \ldots, n\}$	$\rightarrow$	$s_{(n-1)}s_{(1)} = s_{(n)} + s_{(n-1,1)}$
regular representation	$\rightarrow$	$s_{(1)}^n$
$Ind_{S_k\times S_{n-k}}^{S_n} M\otimes N$	$\rightarrow$	$Frob_{S_k}(M)Frob_{S_{n-k}}(N)$
$\bigoplus_{k=0}^n Res_{S_k\times S_{n-k}}^{S_n} M$	$\rightarrow$	$\Delta(Frob_{S_n}(M))$
$Res_{S_k\times S_{n-k}}^{S_n} M$	$\rightarrow$	$\sum_{\lambda\vdash k} s_\lambda^\perp Frob_{S_n}(M)\otimes s_\lambda$
$M\otimes N$ (with S_n acting diagonally)	$\rightarrow$	$Frob_{S_n}(M)\odot Frob_{S_n}(N)$

Adriano also provided me with a construction of the irreducible $Gl_n(\mathbb{C})$ modules. Let T be a standard tableaux of shape λ a partition of k. Let

$$N(T) = \sum_{\sigma\in col(T)} sgn(\sigma)\sigma$$

$$P(T) = \sum_{\sigma\in row(T)} \sigma$$

$$h_\lambda = \text{product of the hooks of } \lambda$$

$col(T)$ and $row(T)$ are the column and row group of the tableau T and are subgroups of S_k. S_k will act on positions of letters in words, that is, a *right* action. $Gl_n(\mathbb{C})$ will have a *left* action on the variables.

Next set

$$E_T = N(T)P(T)/h_\lambda.$$

Now set

$$M^\lambda \simeq \mathcal{L}\{wE_T : w \in [n]^k\}.$$

Note that the elements wE_T form a spanning set and not a basis so one will have to linearly reduce these elements to a basis.

What is interesting to note from this construction is that it is possible to decompose M^λ into submodules of a fixed content. We define the content of a word to be the tuple representing the number of 1s, the number 2s, ..., the number of ns in the word. This tuple is then sorted so that order of the elements does not matter. 0 entries are allowed in this tuple so that the content of a word will be $content(w) = 0^{n_0}1^{n_1}\cdots r^{n_r}$ where $n_0 + n_1 + \cdots + n_r = n$ and $0n_0 + 1n_1 + \cdots rn_r = k$ (e.g. if $n = 4$ then $content(114414) = content(222111) = (0^2 3^2)$). Next we set

$$M^\lambda_\alpha = \{wE_T : w \in [n]^r, content(w) = \alpha\}.$$

Now we have reduced the problem of finding a decomposition of the module M^λ as an S_n module to finding a decomposition of the module M^λ_α for each α. For many special cases of α this is not a difficult problem.

Proposition 3.

$$Frob_{S_{|\lambda|}}(M^\lambda_{1^{|\lambda|}}) = s_\lambda.$$

Proof. This is Schur-Weyl duality.

$$M^\lambda_{1^{|\lambda|}} = \{wE_T : w \in S_n\}$$

□

Proposition 4.

$$Frob_{S_n}(M^\lambda_{0^{n_0}1^{n_1}\cdots r^{n_r}}) = s_{(n_0)} Frob_{S_{n-n_0}}(M^\lambda_{1^{n_1}\cdots r^{n_r}}).$$

Proof. (idea) Show that

$$M^\lambda_{0^{n_0}1^{n_1}\cdots r^{n_r}} \simeq Ind^{S_n}_{S_{n_0}\times S_{n-n_0}} M^\lambda_{1^{n_1}\cdots r^{n_r}}$$

where the action of S_{n_0} is trivial. □

Proposition 5.

$$Frob_{S_n}(M^{(k)}_{0^{n_0}1^{n_1}\cdots r^{n_r}}) = s_{(n_0)} s_{(n_1)} \cdots s_{(n_r)}.$$

Proof. I showed what the decomposition of $M^{(k)}$ was last time using only symmetric function theory. This is (somewhat) a refinement of the statement that

$$Frob_{S_n}(M^{(k)}) = \sum_T s_{\lambda(T)}[X]$$

where the sum is over all column strict tableaux T (non-neg entries less than or equal to n) with content that sums to k.

What I am saying here is that

$$Frob_{S_n}(M^{(k)}_\alpha) = \sum_T s_{\lambda(T)}[X]$$

where the sum is over all column strict tableaux T whose content is α. □

Conjecture 6.

$$Frob_{S_n}(M^{\lambda}_{(1^{|\lambda|-\ell}\ell)}) = s_{(1)} s^{\perp}_{(\ell)} s_{\lambda}$$

Conjecture 7.

$$Frob_{S_n}(M^{\lambda}_{(1^{n_1}2^{n_2}\cdots\ell^{n_\ell})}) = \sum_{\mu\vdash n_1}\sum_{\gamma\vdash k-n_1} c^{\lambda}_{\mu\gamma} Frob_{S_{n-n_1}}(M^{\gamma}_{(2^{n_2}\cdots\ell^{n_\ell})}) s_{\mu}$$

Conjecture 8.

$$Frob_{S_n}(M^{\lambda}_{(d^{n_d}\cdots\ell^{n_\ell})}) = \sum_{\mu\vdash dn_d}\sum_{\gamma\vdash k-dn_d} c^{\lambda}_{\mu\gamma} Frob(M^{\gamma}_{((d+1)^{n_{d+1}}\cdots\ell^{n_\ell})}) Frob(M^{\mu}_{d^{n_d}})$$

This last conjecture is a 'master' conjecture since it implies all the others. Using it we have reduced the calculation from the determination of the decomposition of M^{λ}_{α} to the decomposition of $M^{\lambda}_{(a^b)}$.

I still don't know how to compute $Frob_{S_b}(M^{\lambda}_{(a^b)})$, but I do have the following clues:

Conjecture 9.

$$Frob_{S_2}(M^{\lambda}_{(a^2)}) = \begin{cases} s_{(11)} & \text{if } \lambda = (2a-b, b) \text{ with } b \text{ odd} \\ s_{(2)} & \text{if } \lambda = (2a-b, b) \text{ with } b \text{ even} \\ 0 & \text{else} \end{cases}$$

Conjecture 10.

$$Frob_{S_b}(M^{(1^b)+\lambda}_{(a^b)}) = s_{(1^b)} \odot Frob_{S_b}(M^{\lambda}_{((a-1)^b)})$$

I have a good idea on how to prove most of the conjectures above since their very formulas suggest that there is some module isomorphism that can be used to demonstrate them.

The first cases where one of the conjectures above does not apply is $\alpha = (2^3)$. I was able to compute by process of elimination (since I can compute $Frob(M^{\lambda})$ and $Frob(M^{\lambda}_{\beta})$ for $\beta \neq \alpha$, then we can deduce $Frob(M^{\lambda}_{\alpha})$) that

$$Frob_{S_3}(M^{(33)}_{(222)}) = s_{(1^3)}$$

$$Frob_{S_3}(M^{(42)}_{(222)}) = s_{(2)} s_{(1)}$$

$$Frob_{S_3}(M^{(51)}_{(222)}) = s_{(21)}$$

Any clues about why?

Example: We have by Conjecture 8

$$Frob_{S_4}(M^{(42)}_{(2211)}) = Frob_{S_2}(M^{(4)}_{(22)}) s_{(2)} + Frob_{S_2}(M^{(31)}_{(22)}) s^2_{(1)} + Frob_{S_2}(M^{(22)}_{(22)}) s_{(2)}$$

since $Frob_{S_2}(M^{\lambda}_{(11)}) = s_{\lambda}$.

In addition we know $Frob_{S_2}(M^{(4)}_{(22)}) = 2$ by Proposition 5 and $Frob_{S_2}(M^{(31)}_{(22)}) = s_{(11)}$ and $Frob_{S_2}(M^{(22)}_{(22)}) = s_{(2)}$ by Conjecture 9. Therefore,

$$Frob_{S_4}(M^{(42)}_{(2211)}) = 2s_{(2)}s_{(2)} + s_{(11)}s^2_{(1)}$$

I have placed data for λ a partition of 6 for my conjectures on the decomposition of homogeneous components on the web page for this seminar.