CRM-Fields-PIMS prize: Nicole Tomczak-Jaegermann

Nassif Ghoussoub
Department of Mathematics, University of British Columbia,
Vancouver BC Canada V6T 1Z2
nassif@math.ubc.ca

Nicole Tomczak-Jaegermann, of the University of Alberta, has been awarded the 2006 CRM-Fields-PIMS prize. According to the citation, *"She has made outstanding contributions to infinite dimensional Banach space theory, asymptotic geometric analysis, and the interaction between these two streams of modern functional analysis. She is one of the few mathematicians who have contributed important results to both areas. In particular, her work constitutes an essential ingredient in a solution by the 1998 Fields Medallist W. T. Gowers of the homogeneous space problem raised by Banach in 1932."*

Tomczak-Jaegermann received her Master's (1968) and Ph.D. (1974) degrees from Warsaw University, where she held a position until moving to the University of Alberta in 1983. There she holds a Canada Research Chair in Geometric Analysis. She is a Fellow of the Royal Society of Canada, lectured at the 1998 ICM, and has won the CMSs Krieger-Nelson Prize Lectureship. She has served the Canadian and international research community in many ways, including her current position on the BIRS Scientific Advisory Board and previously as a Site Director of PIMS in Alberta.

What is this area of mathematics which has produced two recent Fields medalists (J. Bourgain and T. Gowers) among many other modern prominent mathematical figures, and yet is still so misunderstood by even the most seasoned of mathematicians? The story starts with the 1932 book of Stefan Banach where he laid the foundation of –infinite dimensional– Banach space theory. It was to be a unifying framework for many problems arising in differential equations and applied fields, but the intellectual curiosity of the customers of the "Scottish cafe" in L'vov took over, and the quest for a "classification theory" for infinite dimensional Banach spaces started soon after. Most problems turned out to be deep and hard and way beyond the reach of the mathematicians of the 30's and 40's. All these questions have now been answered and many solutions had to wait till the end of the century. But while the questions look like mere mathematical curiosities, the techniques developed to answer them turned out to be rich and far reaching: from convex analysis to combinatorics, and from infinite dimensional Ramsey theory, to the refined asymptotics of finite dimensional convex bodies, via the theories of random matrices and of gaussian processes.

Undoubtedly motivated by the structural rigidity of the classical Banach spaces (Hilbert space, L^p-spaces and spaces of continuous functions), S. Banach posed in his book, several intriguing problems about the structure of general infinite dimensional spaces. Are they isomorphic to their own hyperplanes? to their squares or to their cubes? But the most well-known of the lot were undoubtedly *the Schauder basis problem* and *the homogeneous space problem.* Among Nicole Tomczak-Jaegermann's numerous defining contributions to this field, I shall only describe her contributions to these two problems. I will also discuss briefly her more recent work on the *metric entropy.* I will unfortunately not be able to describe her other equally important contributions to *Banach-Mazur distances* between Banach spaces –in particular between the Schatten classes of operators, to her multiple results with H. Koenig [5] of the *best projection constants problem*, her introduction of the seminal concept of *complex convexity* in infinite dimensional complex spaces, her influential paper with A. Pajor [9] on an important strengthening of the so-called *Sudakov's minoration theorem* in the theory of Gaussian processes, as well as her most recent results with S. Szarek discovering the phenomenon of *finite-dimensional saturation* and solving a number of open problems from the early 1980's. For all that, I refer the interested reader to her encyclopedic 1989 monograph [10] and of course to her published work.

Before going into more specifics, it is worth emphasizing that the quest to solve these classical problems has led to a whole new field of study now known as *Asymptotic Geometric Analysis.* Initiated and developed by V. Milman and eventually by many others, this new area of research calls for a deeper understanding of infinite dimensional phenomena via the analysis of various functions of an arbitrarily large number of free variables, as well as certain geometric objects that are determined by an infinitely growing number of parameters. This in turn led to spectacular developments in the so-called *asymptotic theory of convex bodies*, which is roughly concerned with geometric and linear properties of finite-dimensional objects, and the asymptotics of their various quantitative parameters as the dimension tends to infinity.

Results developed in two opposite –yet equally striking– directions. The "optimistic" side was triggered by an early spectacular result of A. Dvoretsky: *every Banach space of sufficiently large dimension contains a subspace that is almost isometric to Hilbert space (ℓ_2^k)* of a given dimension k. In other words, one can find in any n-dimensional convex body a central section of dimension log(n) which is arbitrarily close to a Euclidean ball. This eventually led to a large number of surprising results, the spirit of which being that certain structures get better and better as the dimension grows to infinity. The fact that most of these results can be explained by the *concentration of measure phenomenon* started with the exceptional insight of V. Milman, who subsequently developed the concept further in collaboration with M. Gromov and others (e.g., see [7]) leading to equally remarkable results in geometry and combinatorics. This effort was taken up by M. Talagrand and others in the 90's with great results and striking applications to probability and information theory.

The *pessimistic side* was mostly triggered by Gluskin's result who used probabilistic methods to randomly select certain "pathological" projections of the n-dimensional octahedron (the unit ball in ℓ_1^n). These new objects were then superposed by extremely clever techniques for gluing finite dimensional spaces –initiated by J. Bourgain, S. Szarek, N. Tomczak-Jaegermann and many others– to construct exotic infinite dimensional counterexamples to several long standing problems, some of which are described below.

I. The Schauder basis problem: *Does every Banach space has a basis?*

This problem was of course solved negatively by P. Enflo in the 1970's when he constructed a Banach space without the *approximation property*, and therefore computations in such a space cannot be summarily reduced to manipulating finite dimensional objects, or finite rank operators. In the 1990's, Nicole Tomczak-Jaegermann and her collaborator P. Mankiewicz went way beyond that particular construction, as they developed an ingenuous method to build such counterexamples in a generic way starting from any non-Hilbertian space. They proved the following

Theorem 1 *(N. Tomczak-Jaegermann, P. Mankiewicz [6]) If X is a Banach space not isomorphic to Hilbert space, then $\ell_2(X)$ has necessarily a quotient space which itself contains a subspace with no Schauder basis.*

Recall that if $(X_n)_n$ is a sequence of Banach spaces, their ℓ_2-sum, $(\bigoplus X_n)_{\ell_2}$, is then the Banach space of all sequences of vectors $z = (z_n)$, with $z_n \in X_n$ for all n, such that $\|z\|_{\oplus X_n} = (\sum_n \|z_n\|_{X_n}^2)^{1/2} < \infty$. If $X_n = X$ for all n, we then write $\ell_2(X)$ instead of $(\bigoplus X)_{\ell_2}$.
In other words, spaces without a Schauder basis can now be constructed in just three canonical operations starting from an *arbitrary* Banach space X not isomorphic to Hilbert space. Such spaces are of the form $Z = (\bigoplus Z_n)_{\ell_2}$, where Z_n are *finite-dimensional* quotients of subspaces of $\ell_2(X)$. It should be noted that this theorem is amazingly sharp, in the sense that starting with $\ell_2(X)$ –as opposed to X itself– is necessary, since W.J. Johnson had constructed earlier a Banach space X not isomorphic to Hilbert space, all of whose quotients of subspaces do have a basis.
More remarkable are the techniques used for such a construction. They consist of building infinite-dimensional spaces by properly gluing finite-dimensional ones which are themselves obtained by probabilistic methods for selecting appropriate "random quotients". This line of study was initiated by Gluskin who considered random projections of the n-dimensional octahedron (the unit ball in ℓ_1^n) and proved that the diameter of the Banach-Mazur compactum of n-dimensional normed spaces is of order n. The first one to use finite-dimensional random quotients of ℓ_1^n in an infinite-dimensional construction is J. Bourgain who used it to construct a real Banach space that admits two non-isomorphic complex structures.

II. Banach's homogeneous space problem: *Is Hilbert space the only homogeneous Banach space? i.e., is it the only one that can be isomorphic to all of its infinite dimensional subspaces[1]?*

Now we know that the answer to this question of Banach is affirmative, thanks to independent and remarkably complementary contributions by T. Gowers on one hand, and by N. Tomczak-Jaegerman and her student R. Komorowski on the other. The first obvious difficulty in attacking the homogeneous space problem is the lack of information on the uniform boundedness of norms of the isomorphisms. Even up to this day no direct proof is known of the fact that X being homogeneous, must imply that X is *uniformly* isomorphic to all of its infinite-dimensional subspaces, as is the case for Hilbert space which X is supposed to be after all. However, the breakthrough came when N. Tomczak-Jaegerman and R. Komorowski

[1] Here and throughout, all subspaces are assumed to be closed

proved that much can be said if the space has an *unconditional basis*: that is a basis $\{z_i\}$ such that for some $C > 0$ we have for any scalars $\{a_i\}$, and any choice of signs, $\{\varepsilon_i\}$, that $\| \sum_i \varepsilon_i a_i z_i \| \leq C \| \sum_i a_i z_i \|$.

Theorem 2 *(N. Tomczak-Jaegermann & R. Komorowski [4]) Let X be a Banach space with an unconditional basis, then either X contains a Hilbertian subspace or otherwise it must contain a subspace without an unconditional basis.*

An immediate corollary is the following curious conditional result: If X is a homogeneous Banach space not isomorphic to a Hilbert space, then X cannot have an infinite-dimensional subspace with an unconditional basis. This curiously made a connection with another famous question coming from the 1950s:

Does every infinite-dimensional Banach space has an infinite-dimensional subspace with an unconditional basis?

This question had however received –around the same time– a negative answer by T. Gowers and B. Maurey, via a breakthrough construction that opened a whole new understanding of infinite-dimensional phenomena. This new understanding very fast led to negative solutions for several other problems open for decades, such as *the hyperplane problem* of Banach mentioned above, *the distortion problem* solved by E. Odell and Th. Schlumprecht in [8], as well as many other longstanding open problems. Actually, the Gowers-Maurey space X_0 has a stronger property: no subspace of X_0 is a topological direct sum of two infinite-dimensional Banach spaces. Equivalently, given any two infinite-dimensional subspaces Z and W of X_0, we necessarily have

$$\inf\{\|z - w\|; z \in Z, w \in W, \|z\| = \|w\| = 1\} = 0.$$

That is, the unit spheres of any two infinite dimensional subspaces almost intersect. Such a space X_0 is called *hereditarily indecomposable* (an H.I. space). Moreover, they proved the following.

Theorem 3 *(Gowers-Maurey [3]) A hereditarily indecomposable Banach space is not isomorphic to any proper subspace of itself.*

In other words, these spaces are essentially the counterpart of homogeneous spaces. Note also that an H.I. space cannot have an infinite-dimensional subspace with an unconditional basis, since otherwise, such a subspace would be a direct sum of the span of the even elements of the basis and the span of the odd elements. However, the opposite implication is clearly false since there exist spaces which can be decomposed but still have no subspace with an unconditional basis. Many interesting examples of spaces having these and related properties were eventually constructed by Gowers-Maurey, Odell-Schlumprecht, and Argyros and his co-authors, but the precise connection between subspaces with unconditional basis and H.I. subspaces was finally clarified by the spectacular structural dichotomy proved by Gowers in 1993. In particular, it provided the last missing piece in the solution of the homogeneous space problem.

Theorem 4 *(T. Gowers [2]) Every infinite-dimensional Banach space either has an infinite dimensional subspace with an unconditional basis or has a hereditarily indecomposable subspace.*

The theorem is actually a consequence of a general combinatorial result, which is, in a sense, a vector space analogue of infinite versions of Ramsey theorem.
Once all these results were proved, the solution to the homogeneous space problem is now simple. By the theorem of Tomczak-Jaegermann and R. Komorowsky (Theorem 2), a homogeneous space X not isomorphic to Hilbert space cannot have an infinite-dimensional subspace with an unconditional basis. By Gowers dichotomy theorem, it must contain an H.I. subspace, and hence X itself must be H.I. since it is homogeneous. But then, Theorem 3 of Gowers-Maurey says that it cannot then be isomorphic to any proper subspace of itself, which means that X is not homogeneous after all.

III. The finite-dimensional isomorphic version of the homogenous space problem

It is well known that all finite dimensional Banach spaces of the same dimension (say n) are isomorphic to Euclidean space ℓ_2^n. However, the isomorphism constants can vary wildly, and so one can ask the following finite-dimensional version of the homogenous space problem:

For $0 < \alpha < 1$ and $K \geq 1$ does there exist $f(\alpha, K) > 0$ such that an n-dimensional space X is necessarily $f(\alpha, K)$-isomorphic to Euclidean space ℓ_2^n, whenever all of its $[\alpha n]$-dimensional subspaces are K-isomorphic?

This question is an isomorphic finite-dimensional version of two questions from Banach's book. The first one regards an n-dimensional symmetric convex body all of whose k-dimensional sections are affinely equivalent which was almost completely solved by Gromov in his doctoral thesis ($K = 1$ in the above question). The second one was the homogeneous space problem discussed above.
A positive answer to the above question was proved for sufficiently small α in 1987 by J. Bourgain. In 1989, N. Tomczak-Jaegermann and P. Mankiewicz managed to prove the result for all α, with a "reasonable" function $f(\alpha, K)$. Actually, $f(\alpha, K) \leq cK^{3/2}$ for $0 < \alpha < 2/3$, and cK^2, for $2/3 \leq \alpha < 1$, where c is a constant only depending on α. Both solutions rely again on the study of random quotients of normed spaces already mentioned above. We note that even though the method for constructing specific convex bodies from random projections of polytops, were initiated by Gluskin in 1981, the consideration of random quotients in a general form started with the above results of J. Bourgain and P. Mankiewicz-N. Tomczak-Jaegermann, and its study was eventually developed jointly by the last two authors in a series of papers over the years.

IV. The metric entropy problem

If K and B are two subsets of a vector space (or just a group, or even a homogeneous space), the *covering number* of K by B, denoted $N(K, B)$, is the minimal number of translates of B needed to cover K. Similarly, the *packing* number $M(K, B)$ is the maximal number of disjoint translates of B by elements of K. The two concepts are closely related and we have

$$N(K, B - B) \leq M(K, B) \leq N(K, (B - B)/2).$$

If now B is the unit ball of a normed space and K a subset of that space (the setting and the point of view functional analysts usually employ), these notions reduce to considerations involving the smallest ϵ-nets or the largest ϵ-separated subsets of K.
Besides the obvious geometric framework, packing and covering numbers appear naturally in several fields of mathematics, ranging from classical and functional analysis, through probability theory and operator theory to computer science and information theory (where a code is typically a packing, while covering numbers quantify the complexity of a set). As with other notions related to convexity, an important role is often played by considerations involving duality.
In an operator-theoretic context, one considers the so-called *entropy numbers of an operator* $u : X \to Y$ where X and Y are Banach spaces. They are defined as

$$e_n(u) = \inf\{\varepsilon; \text{ such that } N(uB_X, \varepsilon B_Y) \leq 2^{n-1}\}.$$

These numbers are used to quantify compactness properties of the operator and one can easily see that u is a compact operator if and only if $\lim_n e_n(u) = 0$. Now a classical theorem of Schauder states that u is a *compact operator* if and only if its adjoint u^* is compact, which readily means that the limiting behaviours of the sequences $e_n(u)$ and $e_n(u^*)$ are similar. In 1972, Pietsch asked several specific questions regarding *entropy numbers* and *duality*. Roughly speaking, do these dual entropy numbers always obey similar asymptotic behavious? For example, is it true that $\{e_n(u)\}$ belongs to the space ℓ_p (for some $1 \leq p < \infty$) if and only if $\{e_n(u^*)\}$ does? The strongest version of Pietsch's conjectures can also be formulated in the language of covering numbers in the following way:

There exist numerical constants $a, b \geq 1$ such that for any dimension n and for any two centrally symmetric convex bodies K, B in $\mathbb{R}^n$ one has

$$b^{-1} \log_2 N(B^\circ, aK^\circ) \leq \log_2 N(K, B) \leq b \log_2 N(B^\circ, a^{-1}K^\circ)?$$

Here $A^\circ := \{u \in \mathbb{R}^n : \sup_{x \in A}\langle x, u\rangle \leq 1\}$ denotes the polar body of A.
This conjecture is still open in its full generality. However, the question about the "global" behaviour of entropy numbers was settled positively in 1987 by N. Tomczak-Jaegermann in the special but central case, when either the domain or target space is a Hilbert space, and more generally by J. Bourgain, A. Pajor, S. Szarek and N. Tomczak-Jaegermann in 1989, in the much more general situation where one of the spaces is of type p, for some $p > 1$. Such spaces also comprise all ℓ_p and L_p-spaces (whether classical or non-commutative) for $1 < p < \infty$, as well as all uniformly convex and all uniformly smooth spaces. In this case, the constants a, b depend only on p and they are uniformly bounded if p stays away from 1 and ∞. More recently, the strongest version of Pietsch's conjecture stated above, was established by Artstein, Milman and Szarek in 2003, again in the case when one of the spaces is a Hilbert space (equivalently, when the convex body is an ellipsoid). N. Tomczak-Jaegermann joined effort with them in 2004 (See [1]) to establish the conjecture when one of the spaces is of type $p > 1$, and to develop the theory still further.

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