Introduction to schemes and group schemes

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Contents

1 Sheaves

Definition 1 *Let X be a topological space. Let $Top(X)$ be the category whose objects are the open subsets of X and whose morphisms are the inclusion maps.*[1] *Let Ab be the category of abelian groups. A* **presheaf** *$\mathcal{F}$ of abelian groups on X is a contravariant functor from $Top(X)$ to Ab such that $\mathcal{F}(\emptyset) = 0$.*

Similarly, if C is any category, one can define a presheaf on X with values in C by replacing Ab with C.

Here is some terminology associated to a presheaf $\mathcal{F}$:

[1] This means that $\mathrm{Hom}(V, U) = \emptyset$ if $V \not\subseteq U$, and $\mathrm{Hom}(V, U)$ has just one element if $V \subseteq U$.

- if $U \subseteq X$ is an open set, then the elements of $\mathcal{F}(U)$ are called the **sections** of $\mathcal{F}$ on U; sometimes $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$ or by $H^0(U, \mathcal{F})$;

- if $V \subseteq U \subseteq X$ are open sets, then the morphism $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is called the **restriction map** from U to V; if $s \in \mathcal{F}(U)$, then $\rho_{UV}(s)$ is usually denoted by $s|_V$.

Definition 2 *Let X be a topological space. A* **sheaf** *$\mathcal{F}$ on X is a presheaf satisfying the following additional properties:*

1. *if $U \subseteq X$ is an open set, $(V_i)_i$ is an open covering of U and $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0$ for all i, then we must have $s = 0 \in \mathcal{F}(U)$;*

2. *if $U \subseteq X$ is an open set, $(V_i)_i$ is an open covering of U and $s_i \in \mathcal{F}(V_i)$ are such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all $i \neq j$, then there must exist $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all i.*

Note that s in the second condition above must be unique.

Convention: In this lecture the field K will be algebraically closed, and all the rings will be commutative and unitary.

Examples of sheaves

- $\mathcal{C}^\infty(\mathbb{R})$ is a sheaf.

- $\mathcal{C}^0(\mathbb{R})$ is a sheaf.

- The differentiable maps on a manifold form a sheaf.

- Let K be a field and X/K a variety, endowed with the Zariski topology. For an open set $U \subseteq X$, let
$$\mathcal{O}(U) := \{f : U \longrightarrow K \text{ regular}\}.$$
Then $\mathcal{O}$ is a sheaf, called **the sheaf of regular functions on** X.

- Let A be an abelian group, endowed with the discrete topology. Let X be a topological space. For each $U \subseteq X$ open, let
$$\mathcal{A}(U) := \{f : U \longrightarrow A \text{ continuous}\}.$$
Observe that if U is connected, then $\mathcal{A}(U) \simeq A$. Then $\mathcal{A}$ forms a sheaf, called **the constant sheaf**.

Definition 3 *Let X be a topological space and let $\mathcal{F}$ be a sheaf on X. Let $P \in X$ be a fixed point. We define the* **stalk** *of $\mathcal{F}$ at P by*

$$\mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U),$$

where the direct limit is over open sets $U \subseteq X$ with $P \in U$, taken via the restriction maps.

In other words, an element of $\mathcal{F}_P$ is represented by a pair $\langle U, s \rangle$, where $U \subseteq X$ is an open neighborhood of P and $s \in \mathcal{F}(U)$. Two pairs $\langle U, s \rangle$ and $\langle V, t \rangle$ define the same element of $\mathcal{F}_P$ iff there is an open neighborhood W of P with $W \subseteq U \cap V$ and such that $s|_W = t|_W$. We call the elements of the stalk $\mathcal{F}_P$ **germs** of sections of $\mathcal{F}$ at the point P.

Algebraically, the above definition corresponds to the following. Let K be a field and X/K a variety. Let R_X be the coordinate ring of X. Let $P \in X$ and let m_P be the maximal ideal of P. Then $\mathcal{O}_P = (R_X)_{m_P}$ is the localization of R_X at m_P. Note that $\mathcal{O}$ is the sheaf of regular functions of $X \longrightarrow K$ mentioned on page 2.

Definition 4 *Let X be a topological space and let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A* **morphism of presheaves**

$$\phi : \mathcal{F} \longrightarrow \mathcal{G}$$

consists of morphisms

$$\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for every open set $U \subseteq X$ such that, whenever $V \subseteq U$ is an inclusion, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, then ϕ is called a **morphism of sheaves**. *An* **isomorphism** *is a morphism which has a two-sided inverse.*

In other words, a morphism of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ is a natural transformation if the sheaves $\mathcal{F}, \mathcal{G}$ are considered as functors.

2 Affine schemes

Let K be an algebraically closed field. Let $X \subseteq \mathbb{A}^n_K$ be a variety given by

$$X = \{P \in K^n : f_1(P) = \ldots = f_t(P) = 0\}.$$

Then X corresponds to an ideal

$$I_X := (f \in K[X_1, \ldots, X_n] : f(P) = 0 \; \forall P \in X),$$

which is the radical ideal of $(f_1, \ldots, f_t)$. In other words, we have 1:1 correspondences:

$$X \leftrightarrow \text{ radical ideal } I_X \leftrightarrow \text{ coordinate ring } R_X$$

and

$$\text{points of } X \; \leftrightarrow \; \text{maximal ideals of } R_X.$$

We want to generalize these concepts, as follows.

Definition 5 *Let R be a ring. The* **spectrum** *of R, denoted* $\operatorname{Spec} R$, *is by definition the set of prime ideals P of R, endowed with the following topology: if I is an ideal of R, let*

$$V(I) := \{P : P \in \operatorname{Spec} R, I \subseteq P\}$$

be the closed sets. This is called **the Zariski topology** *of* $\operatorname{Spec} R$.

Definition 6 *Let R be a ring. Let $U \subseteq \operatorname{Spec} R$ be an open set. We define*

$$\mathcal{O}_R(U)$$

to be the set of functions

$$f : U \longrightarrow \amalg_{P \in U} R_P$$

such that

1. $f(P) \in R_P$ *for all* $P \in U$;
2. *for every $P \in U$ there exists an open set $V \ni P$ and there exist $a, b \in R$ such that for every $Q \in V$ we have*

$$f(Q) = \frac{a}{b} \in R_Q \;\; (b \notin Q).$$

Here, R_P is the localization of R at P.

Let us recall that if X/K is a variety, then its regular functions are maps

$$f : X \longrightarrow K$$

such that for every P, there exists an open set $V \ni P$ and there exist polynomials g, h such that $f = \frac{g}{h}$ on V and h never vanishes on V. Thus the definition of regular functions on X/K is a particular case of the above definition of $\mathcal{O}_R$.

Definition 7

1. *A* **ringed space** *is a pair $(X, \mathcal{O})$ consisting of a topological space X and a sheaf of rings $\mathcal{O}$ on X. It is called a* **locally ringed space** *if, in addition, for all $P \in X$, the stalk $\mathcal{O}_P$ is a local ring. The sheaf $\mathcal{O}$ is called the* **structure sheaf** *of the ringed space.*

2. *A* **morphism of ringed spaces** *is a pair*

$$(f, f^\sharp) : (X, \mathcal{O}) \longrightarrow (Y, \mathcal{O}')$$

where

$$f : X \longrightarrow Y$$

is continuous and

$$f^\sharp : \mathcal{O}' \longrightarrow f_* \mathcal{O}'$$

is a morphism of sheaves over Y. If, in addition, for all $P \in X$ the map $f^\sharp$ induces a local ring homomorphism

$$f^\sharp_P : \mathcal{O}'_{f(P)} \longrightarrow \mathcal{O}_P$$

(i.e. the inverse image of the maximal ideal is a maximal ideal), then $(f, f^\sharp)$ is called a **morphism of locally ringed spaces**.

If R is a ring, then $(\operatorname{Spec} R, \mathcal{O}_R)$ is a locally ringed space. These locally ringed spaces are taken as the building blocks to construct schemes.

Definition 8 *An* **affine scheme** *is a locally ringed space $(X, \mathcal{O})$ which is isomorphic (as a locally ringed space) with $(\operatorname{Spec} R, \mathcal{O}_R)$ for some ring R.*

<u>Examples of affine schemes</u>

- Let K be a field. Then $\operatorname{Spec} K$ defines an affine scheme, and consists (as a set) of only one point.

- Let K be an algebraically closed field. We (re)define

$$\mathbb{A}^2_K := \operatorname{Spec} K[X,Y].$$

The **closed points** of $\operatorname{Spec} K[X,Y]$ are in 1:1 correspondence with the maximal ideals of $K[X,Y]$ [which are in 1:1 correspondence with the "usual points" of $\mathbb{A}^2_K$ (defined, say, as in Chapter 1 of [Ha])].

What are the other points of $\operatorname{Spec} K[X,Y]$? They are of two kinds:

  - points of the form $P = (f(X,Y))$ for some irreducible polynomial $f(X,Y) \in K[X,Y]$; these points correspond to the irreducible curves in $\mathbb{A}^2_K$;
  - the point $P' = (0)$, which corresponds to the whole space $\mathbb{A}^2_K$ and which is called the **generic point**.[2]

- Let K be an algebraically closed field and $X \subseteq \mathbb{A}^n_K$ an affine variety of coordinate ring R_X. To X we associate the affine scheme

$$X^{sch} := \operatorname{Spec} R_X.$$

The **closed points** of $\operatorname{Spec} R_V$ are exactly "the points of X" in the usual sense (i.e. the maximal ideals of R_X; they are also called the **geometric points** of X). The other points of $\operatorname{Spec} R_X$ are the irreducible subvarieties of X and the **generic point**, corresponding to the ideal (0). [N.B. The ideal (0) is prime, hence a point of the spectrum, only if X is irreducible. Otherwise X has a generic point for each irreducible component.] Notice that varieties are special affine schemes.

3 Schemes

Let us recall that a projective variety is covered by open subsets which are affine varieties. We introduce a similar concept for schemes:

[2]This is because $V(0) = \{P \supseteq 0\} = \operatorname{Spec} K[X,Y]$, i.e. its closure is everything.

Definition 9 *Let $(X, \mathcal{O}_X)$ be a locally ringed space. It is called a* **scheme** *if for every $P \in X$ there exists an open set $U \ni P$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.*[3]

Definition 10 *A* **morphism of schemes** *is a morphism of locally ringed spaces that are schemes.*

Spec is a functor from the category of rings to the category of schemes, which induces an equivalence of categories between the category of rings and the category of *affine* schemes.

Note that not every scheme is projective. We have 1:1 correspondences:

$$\text{affine variety } X \leftrightarrow R_X \leftrightarrow \operatorname{Spec} R_X,$$

$$\text{projective variety } X \leftrightarrow R_X \leftrightarrow \operatorname{Proj} R_X,$$

where $\operatorname{Proj} R_X$ has the same construction as $\operatorname{Spec} R_X$, but it takes into account only homogeneous objects, as follows:

Definition 11 *Let $R = \oplus_{d \geq 0} R_d$ be a graded ring, and let $R_+ := \oplus_{d>0} R_d$.* [4] *Let*

$$\operatorname{Proj} R := \{P : P \textit{ homogeneous prime ideal of } R, R_+ \not\subseteq P\}.$$

Additionally, for $U \subseteq \operatorname{Proj} R$, let

$$\mathcal{O}_R(U)$$

be the set of functions

$$f : U \longrightarrow \amalg_{U \ni P} R_{(P)}$$

such that

1. *$f(P) \in R_{(P)}$ for all $P \in U$;*

2. *for every $P \in U$ there exists an open set $V \ni P$ and there exist $a, b \in R$ such that for every $Q \in V$ we have*

$$f(Q) = \frac{a}{b} \in R_{(Q)} \quad (b \notin Q),$$

where $R_{(P)}$ denotes the degree zero elements of $T^{-1}R$ with T all homogeneous elements in $R \backslash P$.

The pair $(\operatorname{Proj} R, \mathcal{O}_R)$ is a scheme.

<u>Examples of schemes</u>

[3]As before, if $V \subseteq X$ is an open set, then $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V \cap U)$.

[4]One can think of $R = K[X, Y]$ and $R_+ = (X, Y)$.

- $\mathbb{P}^n_K = \operatorname{Proj} K[X_0, \ldots, X_n]$ is a scheme.
- $\mathbb{P}^n_R = \operatorname{Proj} R[X_0, \ldots, X_n]$ is a scheme.
- We have 1:1 correspondences:

$$\text{projective schemes in } \mathbb{P}^n_K \leftrightarrow K[X_0, \ldots, X_n]/I, \text{ where } I \text{ is homogeneous saturated}$$ [5]

$$\text{projective varieties in } \mathbb{P}^n_K \leftrightarrow K[X_0, \ldots, X_n]/I, \text{ where } I \text{ is homogeneous radical}$$

<u>Example of a point and a double point</u>

Let K be an algebraically closed field. Let $a, b \in K$. Then:

- $I = (X - a, Y - b)^2$ is a scheme, but not a variety;
- $J = (X - a, Y - b)$ is a variety;
- $L = \big((X - a)^2, Y - b\big)$ is a scheme, but not a variety.

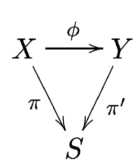
The underlying sets of I, J, L are the same –they consist of only one point, (a, b); however, for I, the point is "thick" in all directions; for J the point is "small"; for L the point comes with a "tangent direction", namely $y - b$. Algebraically, the only linear equation in the ideal L is $y - b$ (this happens e.g. when we intersect a plane curve with a tangent line that has intersection multiplicity 2 with the curve).

4 S-schemes

Definition 12 *Let X, S be schemes and let $\pi : X \longrightarrow S$ be a morphism of schemes. Then X is called an* **S-scheme***.*

If X is an S-scheme, we think of X as "a family of schemes parametrized by S".

Definition 13 *Let S be a scheme and $\pi : X \longrightarrow S$, $\pi' : Y \longrightarrow S$ S-schemes. A map $\phi : X \longrightarrow Y$ is called a* **morphism of S-schemes** *(or S-morphism) if it is a morphism of schemes such that $\pi' \circ \phi = \pi$, i.e. the following diagram is commutative:*



We have thus defined a category which generalizes the category of schemes over K, which corresponds to the case $S = \operatorname{Spec} K$. Also note that every scheme is a $\operatorname{Spec} \mathbb{Z}$-scheme (because every ring R admits a unique homomorphism $\mathbb{Z} \longrightarrow R$).

Definition 14 *Let S be a scheme and X, Y two S-schemes. We define a* **T-valued point of X** *to be an S-morphism $T \longrightarrow X$. We denote by*

$$X(T) := \operatorname{Hom}_S(T, X) = \{S - \mathit{morphisms}\ T \longrightarrow X\}$$

the set of all T-valued points of X. If $T = S$, we call $X(S)$ the set of **sections** *of the S-scheme X. If $S = T = \operatorname{Spec} R$ for some ring R, we denote $X(S) = X(R)$.*

In other words, given an S-scheme X, we have a contravariant functor

$$\text{category of } S\text{-schemes} \longrightarrow \text{category of sets}$$

$$T \mapsto X(T).$$

Let us note that if X is a scheme and $P \in X$ is a point, then P defines a local ring $\mathcal{O}_P$, the stalk of the structure sheaf at P, and hence a maximal ideal m_P and a residue field $k(P) := \mathcal{O}_P/m_P$. Using these notions, we can introduce the concept of X being regular at P.

Definition 15 *A scheme X is called* **reduced** *if the rings of its structure sheaf contain no nilpotent elements.*

A scheme X is called **irreducible** *if its associated topological space is irreducible.*

A scheme X is called **integral** *if it is both reduced and irreducible.*

Definition 16 *Let X be a scheme.*

1. *If X is irreducible, then the* **dimension** *of X is the maximal length n of a chain of distinct irreducible closed subsets*

$$X_0 \subset X_1 \subset \ldots \subset X_n = X.$$

2. *If X is arbitrary, then the* **dimension** *of X is the maximal dimension of its irreducible components.*

3. *Let $p \in X$ be a point. We say that X is* **regular at** *P if*

$$\dim X = \dim_{k(P)}(m_P/m_P^2).$$

5 Group schemes

Definition 17 *A* **group scheme** *is a scheme which is a group such that the group operations and the inversion map are morphisms of schemes.*

Definition 18 *Let $\pi : X \longrightarrow S$ be an S-scheme. X is called a* **group scheme over** *S if there exists an S-morphism*

$$e : S \longrightarrow X$$

(the identity) and there exist S-morphisms

$$\mu : X \times X \longrightarrow X,$$

$$\rho : X \longrightarrow X$$

(the operation and the inversion) satisfying:

1. $\mu \circ (\mathrm{id} \times \rho) = e \circ \pi$,

2. $\mu \circ (\mu \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mu)$,

i.e. the following diagrams are commutative:

$$\begin{CD} X \times X @>{\mathrm{id} \times \rho}>> X \times X \\ @V{\pi}VV @VV{\mu}V \\ S @>>{e}> X \end{CD}$$

$$\begin{CD} X \times X \times X @>{\mu \times \mathrm{id}}>> X \times X \\ @V{\mathrm{id} \times \mu}VV @VV{\mu}V \\ X \times X @>>{\mu}> X \end{CD}$$

Let us note that a group scheme over S is a group object in the category of S-schemes.

<u>Examples of group schemes</u>

- $\mathbb{G}_a \simeq \mathbb{A}^1_K$ is a group variety over $\operatorname{Spec} K$.

- $\mathbb{G}_m \simeq \mathbb{A}^1_K \backslash \{0\}$ is a group variety over $\operatorname{Spec} K$.

- A group variety X/K is a group scheme over $\operatorname{Spec} K$.

- An elliptic curve is a group variety.

- The Jacobian of a smooth curve is a group variety.

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