Equilibria in a Sorting Problem

Li, Hao

University of Toronto

March 12, 2008

Prepared for seminar at Fields Institute, University of Toronto

Competing for Talents, Damiano, Li and Suen, 2005

First in Village or Second in Rome? Damiano, Li and Suen, 2008

Model: Agents

- Agents of mass of 2 must each choose one of the two organizations, A and B.
  - Each agent is associated with a type θ.
  - θ is continuously distributed on $[0, 1]$.
- Agents care about the amount of resources $R_i(\theta)$ they receive as a member of the organization they join (pecking order effect), and the mean type m_i of the organization they join (peer effect), $i = A, B$.
  - Payoff to an agent of type θ from joining organization $i = A, B$ is $V(R_i(\theta), m_i)$, with V continuous and increasing in each argument.

Model: Organizations

- Capacity constraint: each organization i admits a mass 1 of agents.
- Resource constraint: each organization has a fixed budget Y_i of resources that it distributes among its members according to their rank within the organization.
- Each organization i chooses a resource distribution schedule $S_i : [0, 1] \to \mathbb{R}_+$.
  - For each $r \in [0, 1]$, $S_i(r)$ is amount of resources received by an agent whose quantile rank is r in organization i.
- Set of admissible resource distribution schedules for i: $\mathcal{S}_i$
  - non-negative resources only: S_i is non-negative;
  - meritocracy: S_i is non-decreasing;
  - resource constraint: $\int_0^1 S_i(r)\,\mathrm{d}r \leq Y_i$;
  - technical condition: S_i is almost everywhere continuously differentiable.
- Organization's objective: each organization i maximizes its own quality m_i, the average type of its members.

Model: Timing

- Resource distribution stage: A and B simultaneously choose resource distribution schedules S_A and S_B.
- Sorting stage: given (S_A, S_B), sorting equilibrium determines allocation of agents and payoffs to organizations.
- A two-stage game

Model: Feasible allocations at sorting stage

- A feasible allocation is a pair (H_A, H_B) of cumulative type distributions in organizations A and B such that

$$H_A(\theta) + H_B(\theta) = 2\theta \text{ for all } \theta \in [0, 1].$$

- We are implicitly restricting attention to allocations where all agents join one organization.
  - Joining any organization is better than not joining either.
- The rank of an agent θ in each organization depends on the allocation (H_A, H_B):

$$r_A(\theta) = H_A(\theta);$$

$$r_B(\theta) = H_B(\theta).$$

- Average type $m_i = \int \theta \, \mathrm{d}H_i(\theta)$.
- Payoff of type θ from joining organization i is $V(S_i(r_i(\theta)), m_i)$.

Model: Definition of sorting equilibrium

- Given an allocation, (H_A, H_B), an agent θ assigned to i has the option to move to organization j, if the agent's type is higher than the lowest type in organization j.
- Definition 1. Given resource distribution schedules (S_A, S_B), a sorting equilibrium is a feasible allocation (H_A, H_B) such that if H_i is strictly increasing on (θ, θ') and $H_j(\theta) > 0$, then $V(S_i(r_i(\theta)), m_i) \geq V(S_j(r_j(\theta)), m_j)$.
  - H_i increasing on (θ, θ') means that agents of types in this interval are joining i.
  - If $H_j(\theta) > 0$, these agents could move to j.

Model: Existence and selection of sorting equilibrium

- Each sorting equilibrium (H_A, H_B) is associated with a fixed point of the mapping from the set of possible m_A's to itself.
  - The mapping is monotone increasing so a fixed point exists.
- Multiple sorting equilibria may exist.
- We select on the "A-dominant sorting equilibrium" with the largest m_A.
  - For any (S_A, S_B), let $T_A(S_A, S_B)$ be the value of $m_A - m_B$ in the A-dominant sorting equilibrium.

Model: Resource distribution game

- Players: A and B
- Strategies: each organization, $i = A, B$ independently chooses a resource distribution schedule $S_i \in \mathcal{S}_i$.
- Payoffs: for each strategy profile (S_A, S_B) the payoff to i is its average type m_i in the A-dominant equilibrium.
- Since $m_A + m_B$ is a constant, the resource distribution game is strictly competitive.
- A strategy profile (S_A^*, S_B^*) is a Nash equilibrium of resource distribution game if and only if

$$S_A^* \in \arg \max_{S_A \in \mathcal{S}_A} \min_{S_B \in \mathcal{S}_B} T_A(S_A, S_B);$$

$$S_B^* \in \arg \min_{S_B \in \mathcal{S}_B} \max_{S_A \in \mathcal{S}_A} T_A(S_A, S_B);$$

$$\max_{S_A \in \mathcal{S}_A} \min_{S_B \in \mathcal{S}_B} T_A(S_A, S_B) = \min_{S_B \in \mathcal{S}_B} \max_{S_A \in \mathcal{S}_A} T_A(S_A, S_B).$$

Motivation

- Why is the sorting problem interesting to applied economists?
- Understanding coexistence of mixing and segregation in the distribution of talents among organizations.
  - At two levels: comparative statics analysis of sorting equilibrium; "endogenizing" the pecking order effect.
- Providing policy recommendations
  - Education policy: Texas ten-percent law
- The problem is difficult.
  - Strategy space is too large.
  - Sorting equilibrium is difficult to characterize for arbitary pair (S_A, S_B).

Solution Method

- First solve the problem

$$\min_{S_B \in \mathcal{S}_B} \max_{S_A \in \mathcal{S}_A} T_A(S_A, S_B).$$

- Find solution to above problem in two steps:
  - Fix some $S_B \in \mathcal{S}_B$ and a target T of average type difference $m_A - m_B$. Find the minimum budget $C(T; S_B)$ of resources for A such that T is attained in a sorting equilibrium.
  - Then, let B choose S_B to maximize the resulting minimum budget function $C(T; S_B)$ subject to the budget constraint for B.
- If for any $T \geq 0$ there exists some S_B^* that solves the above two-step problem, then the minmax value of the game is given by T^* such that $C(T^*; S_B^*) = Y_A$.

Key Simplifications

- Assume V takes a linear form: $V(R_i(\theta), m_i) = \alpha R_i(\theta) + m_i$, for some positive constant α.
- Assume type distribution is uniform on $[0, 1]$.
- The above assumptions are sufficient to reduce the step of finding the minimum budget function $C(T; S_B)$ to a linear programming problem.
  - For this presentation, I assume $Y_A = Y_B = Y$.

Preliminary Analysis: Quantile-quantile plot

- To each allocation (H_A, H_B) we can associate a function $t : [0,1] \to [0,1]$ defined as

$$t(r) \equiv 1 - H_A\left(\inf\{\theta : H_B(\theta) = r\}\right) \quad \forall r \in [0,1].$$

  - $t(r)$ is the fraction of agents in organization A of type higher than rank r's type in organization B.

Preliminary Analysis: Quality difference

- Under uniform type distribution, average type difference $m_A - m_B$ in any allocation is directly related to an integral of the function t.
  - Lemma. Let (H_A, H_B) be a feasible allocation and t the associated quantile-quantile plot. Then

$$m_A - m_B = -\frac{1}{2} + \int_0^1 t(r)\, \mathrm{d}r.$$

- For any function t we refer to $\int_0^1 t(r)\, \mathrm{d}r = T$ as the "quality difference."
  - Possible quality differences: $T \in [0, 1]$.

Preliminary Analysis: Quality premium

- Under linear payoff function, there is a simple characterization of sorting equilibrium in terms of quantitle-quantile plot.
- Given $S_B(r)$, a quantile-quantile plot t such that $\int_0^1 t(r)\,\mathrm{d}r = T$ corresponds to a sorting equilibrium for S_A^t such that

$$S_A^t(1 - t(r)) = S_B(r) - P(T) \quad \text{for all } r \in [0, 1].$$

  - $P(T) = (T - 1/2)/\alpha$ is quality premium.

Minmax Value: Expenditure minimization

- For fixed S_B, and each $T \geq 1/2$ what is the "cheapest" S_A such that for S_B and S_A, there is a sorting equilibrium with quality difference T?
- We can write the minimization problem as

$$C(T; S_B) \equiv \min_t \int_0^1 S_A^t(r)\ \mathrm{d}r \quad \text{s.t.} \int_0^1 t(r)\ \mathrm{d}r = T.$$

- By the definition of S_A^t, $C(T; S_B)$ is given by

$$\min_{t: \int_0^1 t(r)\ dr = T} \int_{t^{-1}(0)}^{t^{-1}(1)} t(r)\Delta_B'(r)\ \mathrm{d}r.$$

  - $\Delta_B(r) = \max\{S_B(r) - P(T), 0\}$ is B's effective schedule.
- We have a linear programming problem.

- Lemma. There is a solution in $t(r)$ to the expenditure minimization problem that assumes a countable number of values.
  - When S'_B is decreasing, a constant $t(r)$ reduces expenditure.
  - When S'_B is increasing, a step function $t(r)$ reduces expenditure.
- Lemma. There exists a solution $t(r)$ to the expenditure minimization problem that assumes at most one value strictly between 0 and 1.
  - Since t is a step function, the objective function is linear in the value t assumes in between any two discontinuity points.

Minmax Value: Expenditure minimization

- Solution to the expenditure minimization problem can be explicitly characterized.
- Solution $t(r)$ is fully characterized by discontinuity point(s) and the resource constraint.
    - For allocation functions with two discontinuity points, r^1 and r^0, we have $t(r) = (T - r^1)/(r^0 - r^1)$ for $r \in (r^1, r^0)$.
    - For allocation functions with one discontinuity point, $\hat{r}$, we have $t(r) = T/\hat{r}$ for $r \leq \hat{r}$.
- $C(T; S_B)$ is given by the minimum of the solution to

$$\min_{r \geq T} \frac{T}{r} \Delta(r)$$

and the solution to

$$\min_{T \geq r^1 \geq 0;\ 1 \geq r^0 \geq T} \Delta_B(r^1) + \frac{T - r^1}{r^0 - r^1} \left(\Delta(r^0) - \Delta_B(r^1) \right).$$

Minmax Value: Budget maximization problem

- For each T, consider the problem of B choosing S_B to maximize $C(T; S_B)$

$$C(T) \equiv \max_{S_B} C(T; S_B) \quad \text{s.t} \int_0^1 S_B(r)\, \mathrm{d}r \le Y.$$

- The characterization of the solution the expenditure minimization problem implies that the solution $S_B(r)$ takes the form

$$S_B(r) = \begin{cases} 0 & \text{if } r < \tilde{r}; \\ P(T) + \beta(r - \tilde{r}) & \text{if } r \ge \tilde{r}, \end{cases}$$

where β is a constant determined by the resource constraint and $\tilde{r}$.

Minmax Value: Budget function

- Lemma. $C(1/2) = Y$, and $\lim_{T \to 1} C(T) = \infty$. Moreover:
  - if $\alpha Y > 1/2$, then $C'(T) > 0$ for all $T \geq 1/2$;
  - if $\alpha Y \in [1/16, 1/2]$, then there is a $\hat{T}$ such that $C'(T) < 0$ for $T \in (1/2, \hat{T})$ and $C'(T) > 0$ for $T \in (\hat{T}, 1)$;
  - if $\alpha Y < 1/16$, then there exist T_- and T_+ such that $C'(T) < 0$ for $T \in (1/2, T_-)$; $C'(T) > 0$ for $T \in (T_+, 1)$ and $C(T) = 0$ for $T \in [T_-, T_+]$.

Minmax Value: Minmax quality difference

- Let T^* be the largest T for which $C(T) = Y$.
- T^* is a lower bound on the minmax value.
  - Since $C(T^*) = Y$, for any S_B there exists an S_A which respects the resource constraint such that given (S_A, S_B) there is a sorting equilibrium with quality difference at least T^*.
- Let S_B^* be the maximizer of $C(T^*; S_B)$.
  - T^* is the minmax value because $C(T; S_B^*) > Y$ for all $T > T^*$.

Minmax Value: Minmax strategy

- Proposition. Let $T^* = \max\{T \in [1/2, 1] : \; C(T) = Y\}$. The minmax problem $\min_{S_B \in \mathcal{S}} \max_{S_A \in \mathcal{S}} T_A(S_A, S_B)$ admits a unique solution S_B^*, given by

$$S_B^* = \begin{cases} 0 & \text{if } r < r^*; \\ P(T^*) + \beta(r - r^*) & \text{if } r \geq r^* \end{cases}$$

where

$$r^* = 1 - \frac{2}{1/(1 - T^*) + P(T^*)/Y};$$
$$\beta = \frac{2(Y - P(T^*)(1 - r^*))}{(1 - r^*)^2}.$$

Nash Equilibrium: Existence

- To construct a Nash equilibrium we note that there are many A best responses to S_B^*.
- Lemma. Let S_A be a resource allocation schedule such that $S_A(1) \leq S_B^*(1) - (T^* - 1/2)/\alpha$ and $\int_0^1 S_A(r)\, \mathrm{d}r = Y$. Then S_A is a best response to S_B^*.
- Proposition. Let $\hat{r} = P(T^*)/S_B^*(1)$ and S_A^* be defined by

$$S_A^*(r) = \begin{cases} 0 & \text{if } r < \hat{r}; \\ (r - \hat{r})S_B^*(1) & \text{if } r \geq \hat{r}. \end{cases}$$

Then, the strategy profile (S_A^*, S_B^*) is a Nash equilibrium of the resource distribution game.

Nash Equilibrium: Resource distributions and sorting structure

- Competition at the top
  - only high ranks receive positive resources in equilibrium;
  - resources strictly increase with rank above a threshold;
  - rate of increase is slower in the dominant organization.
- Mixing at the top
  - top talents are found in both organizations;
  - top talents are found in larger number in the dominant organization.

Nash Equilibrium: Comparative statics

- If the peer effect becomes less important (an increase in α or Y):
  - T^* and r^* both decrease;
  - B competes on a larger set of ranks;
  - more mixing at the top;
  - B's share of top talents increases;
  - flatter resource distribution schedules.