Some Theorems in Math I

and Physics I revisited

The Schroedinger operator

$$S_h = -h^2 \left(\frac{d}{dx}\right)^2 + V(x)$$

In this talk we'll assume

$$V \in C^\infty(\mathbb{R})$$

and

$$V(x) \to +\infty \quad \text{as} \quad |x| \to \infty$$

②

Theorem S_h has discrete spectrum

$$\lambda_1(h) \leq \lambda_2(h) \leq \cdots$$

with $\lambda_i(h) \to +\infty$ as $i \to \infty$

Example The harmonic oscillator : $V(x) = x^2$

$$\lambda_i(h) = h(2i+1)$$

The Weyl law:

Let $H(x, \xi) = \xi^2 + V(x)$

<u>Theorem</u> (Weyl) As $h \to 0$

$$\# \{ \lambda_i(h) < \lambda \} \sim \frac{1}{2\pi h} \, \text{area}(H < \lambda)$$

<u>Example</u> $V(x) = x^2$

$$\# \{ \lambda_i(h) < \lambda \} \sim \frac{1}{2\pi h} (\pi \lambda) = \frac{\lambda}{2h}$$

Inverse results

Suppose $V(x) = V(-x)$

and $V(x)'' > 0$ (i.e V

strictly convex.)

Theorem For arbitrary $h_0 > 0$,

$\mathrm{Spec}(S_h)$, $0 < h \leq h_0$, determines

V.

Remark

Our goal is not only to prove this but to give an explicit way of recapturing V from the Weyl asymptotics

Proof We can assume without loss of generality that $V(0) = 0$

Let

$$A(\lambda) = \text{area}(H \leq \lambda)$$

and note that

$$H(x, \xi) \leq \lambda \iff |\xi| \leq \sqrt{\lambda - V(x)}$$

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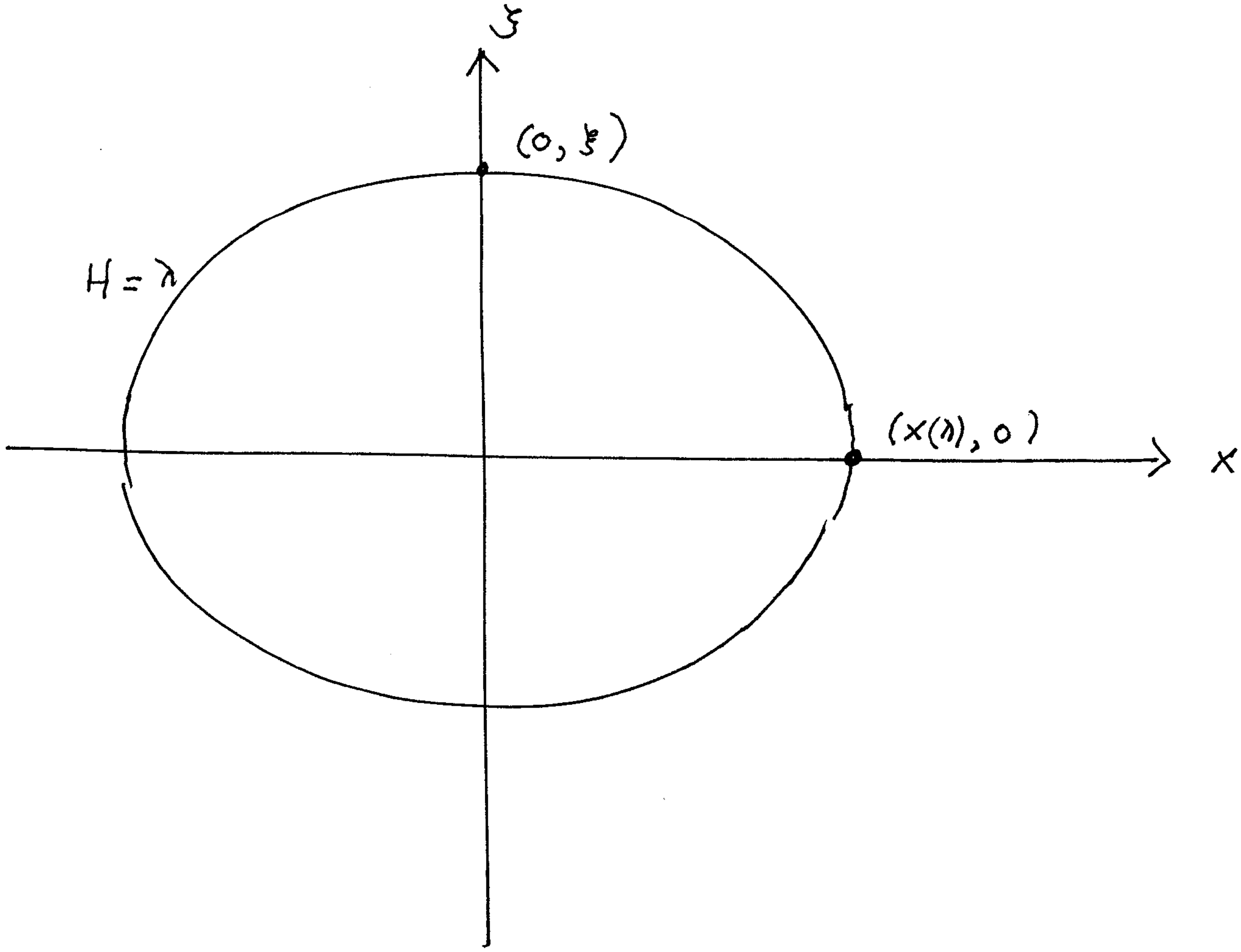

$$H(x, \xi) = H(\pm x, \pm \xi)$$

Thus

$$A(\lambda) = 4 \int_0^{x(\lambda)} \sqrt{\lambda - V(x)}\, dx$$

where $V(x(\lambda)) = \lambda$

For $t \geq 0$ let $f(t) = V^{-1}(t)$

i.e.

$$f(t) = x \iff t = V(x)$$

(8)

Then $f(\lambda) = x(\lambda)$ and

$V(f(t)) = t$ so

$$\boxed{A(\lambda) = 4 \int_0^{\lambda} (\lambda - t)^{\frac{1}{2}} f'(t)\, dt}$$

Fractional integration

For $a > 0$ and $g \in \mathcal{L}^{\infty}([0,\infty))$,

$$J^a g(\lambda) \overset{\text{def}}{=} \frac{1}{\Gamma(a)} \int_0^{\lambda} (\lambda - t)^{a-1} g(t)\, dt$$

Properties

$$J^a J^b = J^{a+b}$$

$$J^1 g(\lambda) = \int_0^{\lambda} g(t)\, dt$$

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Thus for $f(t) = V^{-1}(t)$

$$A(\lambda) = \left(4\,\Gamma\left(\tfrac{3}{2}\right) J^{3/2} f'\right)(\lambda)$$

$$= 2\sqrt{\pi}\left(J^{3/2} f'\right)(\lambda) = 2\sqrt{\pi}\, J^{\frac{1}{2}} f(\lambda)$$

so

$$V^{-1} = f = \frac{1}{2\sqrt{\pi}}\left(J^{\frac{1}{2}} A\right)'$$

<u>Remark</u> The classical version of this result goes back to Abel:

Consider the classical mechanical system on $\mathbb{R}^2$ defined by the Hamilton - Jacobi equations

$$\dot{x}(t) = \frac{\partial H}{\partial \xi}$$

$$\dot{\xi}(t) = -\frac{\partial H}{\partial x}$$

The integral curves of this equation lie on the level sets,

$H = \lambda$.

Let $T(\lambda)$ be the <u>period</u> of the integral curve lying on $H = \lambda$, i.e. the time it takes for a classical particle to go around this level set and come back to its starting point.

By Hamilton - Jacobi :

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\xi^2 + V(x)\right) = 2\xi \text{ ; so}$$

$$T(\lambda) = \int_{H=\lambda} dt = \int_{H=\lambda} \frac{1}{2\xi}\frac{dx}{dt}\,dt$$

$$= \int_{H=\lambda} \frac{dx}{2\xi}$$

$$= 4\int_0^{x(\lambda)} \frac{dx}{2\sqrt{\lambda - V(x)}}$$

where $\lambda = V(x(\lambda))$

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Let $f(t) = V^{-1}(t)$

Then

$$T(\lambda) = 2 \int_0^\lambda (\lambda - s)^{-\frac{1}{2}} f'(s)\, ds$$

$$= 2\pi^{\frac{1}{2}} \left(J^{\frac{1}{2}} f'\right)(\lambda)$$

so

$$\left(J^{\frac{1}{2}} T\right)(\lambda) = 2\pi^{\frac{1}{2}} f(\lambda)$$

Note that, by our previous result,

$$A(\lambda) = \text{area}(H \leq \lambda) = 2\pi^{\frac{1}{2}} J^{\frac{1}{2}} f(\lambda)$$

$$= J^{\frac{1}{2}}(J^{\frac{1}{2}} T)(\lambda)$$

$$= JT(\lambda) = \int_0^{\lambda} T(t)\,dt$$

verifying, for Hamiltonians of the type above, a general fact:

$$\frac{dA(\lambda)}{d\lambda} = T(\lambda)$$

A local version of the inverse result above:

Assume

a) $V(0) = 0$

b) $V(x) = V(-x)$ near $x_0 = 0$

c) $V''(x) > 0$ near $x_0 = 0$

d) If $V(x_0) = V'(x_0) = 0$

then $x_0 = 0$

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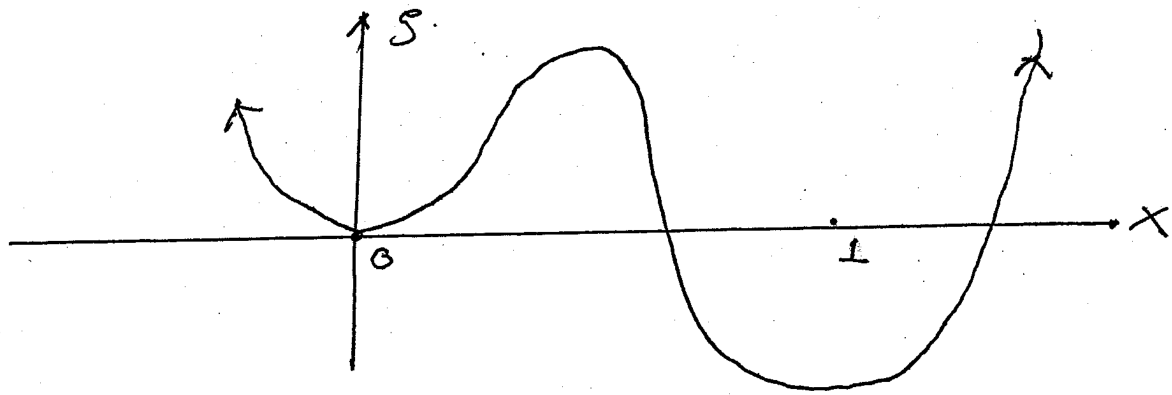
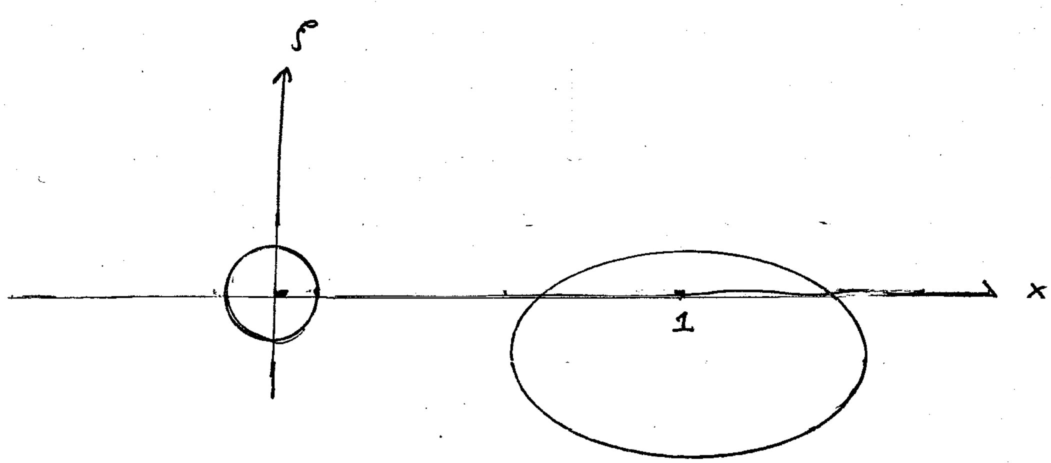
For instance



The set $H(x,\xi) \le \lambda$, $0 < \lambda < \varepsilon$

in this example



$$\text{Area}\,(H \le \lambda) = A_0(\lambda) + A_1(\lambda)$$

As $\lambda \to 0$ the left hand region shrinks to a point and then disappears. Thus, for $-\varepsilon < \lambda < 0$,

$$\text{Area}(H \le \lambda) = A_1(\lambda)$$

or, in other words, for $-\varepsilon < \lambda < \varepsilon$

$$\text{area}(H \le \lambda) = A_0(\lambda)\, 1_+(\lambda) + A_1(\lambda)$$

where $A_0, A_1 \in C^\infty(-\varepsilon, \varepsilon)$

<u>Conclusion</u> The Weyl asymptotics of S_h determines the Taylor series of $A_0(\lambda)$ at $\lambda = 0$ and hence the Taylor series of $V(x)$ at $x = 0$

As motivation for what I'll be doing in my second lecture, I'll describe another way of looking at these results: via Birkhoff canonical forms

Lets assume as above that

$V(0) = V'(0) = 0$ and $V''(x) > 0$

but lets <u>not</u> assume $V(x) = V(-x)$

As above let

$$H(x,\xi) = \xi^2 + V(x)$$

Theorem There exists a function,

$H_{BC} \in C^\infty(\mathbb{R})$ and an area-preserving

diffeomorphism,

$$\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$$

such that

$$\phi^* H_{BC}(x^2 + \xi^2) = H.$$

Sketch of the proof

Suppose H_{BC} and α exist.

Then

$$A(\lambda) = \text{area}(H \leq \lambda) = A_{BC}(\lambda)$$

$$= H_{BC}^{-1}(\lambda)\,\pi$$

since $H_{BC} \leq \lambda \iff x^2 + \xi^2 \leq H_{BC}^{-1}(\lambda)$

Thus $H_{BC}^{-1} = \frac{1}{\pi} A$.

Let

$$(*) \qquad \dot{x} = \frac{\partial H}{\partial \xi}, \quad \dot{\xi} = -\frac{\partial H}{\partial x}$$

and

$$(*)_{BC} \qquad \dot{x} = \frac{\partial H_{BC}}{\partial \xi}, \quad \dot{\xi} = -\frac{\partial H_{BC}}{\partial x}$$

be the Hamiltonian systems associated with H and H_{BC}.

By the area-period relation

$$T(\lambda) = \frac{\partial A}{\partial \lambda} = \frac{\partial}{\partial \lambda} A_{BC} = T_{BC}(\lambda)$$

Thus the map, α, is completely specified if we require that it map

(a) the level set $H^{-1}(\lambda)$ onto the level set $H_{BC}^{-1}(\lambda)$

(b) the integral curves of $(*)$ onto the integral curves of $(*)_{BC}$

(c) the positive x-axis onto itself.

Notice that the condition (b) makes sense in view of the area-period relation: It takes the same time for a point-mass satisfying $(*)$ to go around the curve, $H^{-1}(\lambda)$, as it does for a point-mass satisfying $(*)_{BC}$ to go around the curve, $H_{BC}^{-1}(\lambda)$.

In the course of proving the theorem above we showed that

$$\pi \, H_{BC}^{-1}(\lambda) = \text{area}\,(H \le \lambda) = A(\lambda)$$

We also proved earlier that if

$$V(x) = V(-x) \qquad \text{then :}$$

$$A(\lambda) = 2\sqrt{\pi}\, J^{\frac{1}{2}} f(\lambda)$$

where $f = V^{-1}(\lambda)$ for $0 \le \lambda < \infty$

Thus if $V(x) = V(-x)$

$$H_{BC}^{-1}(\lambda) = \frac{2}{\sqrt{\pi}}\left(J^{\frac{1}{2}}V^{-1}\right)(\lambda)$$

This formula gives one an explicit way of recapturing V from H_{BC} and vice-versa when

$V(x) = V(-x)$

with trivial modifications

the argument I just sketched

works at local minima of

V:

<u>Theorem</u> If $V(0) = V'(0) = 0$

and $V''(0) > 0$ there exists

a neighborhood, $\mathcal{U}$, of $x = \xi = 0$,

a function, $H_{BC} \in C^{\infty}(-\varepsilon, \varepsilon)$

and an area preserving diffeomorphism,

φ, of $\mathcal{U}$ onto the disk, $x^2 + \xi^2 < \varepsilon$

such that $\varphi^* H_{BC}(x^2 + \xi^2) = H$

We can also interpict our earlier result about what happens at a local minimum, $V(0)=0$, of V as saying:

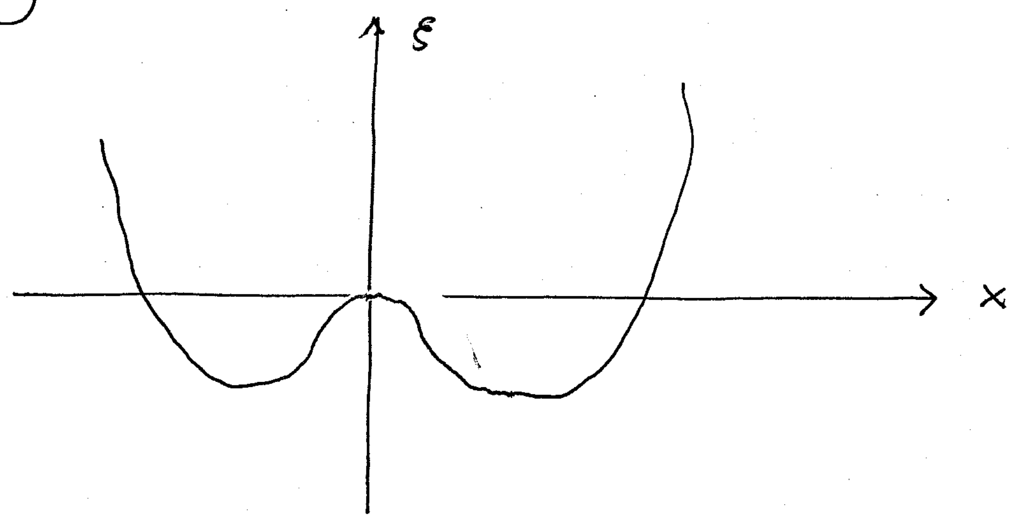
<u>Theorem</u> Let $A(\lambda)$ be the area of the set, $\xi^2 + V(x) = H(x,\xi) < \lambda$,

Then for $-\varepsilon < \lambda < \varepsilon$

$$A(\lambda) = \pi H_{BC}^{-1}(\lambda)\, 1_+(\lambda) + F(\lambda)$$

with F in $C^\infty(-\varepsilon,\varepsilon)$. In particular the Weyl asymptotics determines the Taylor series of H_{BC} at $\lambda = 0$

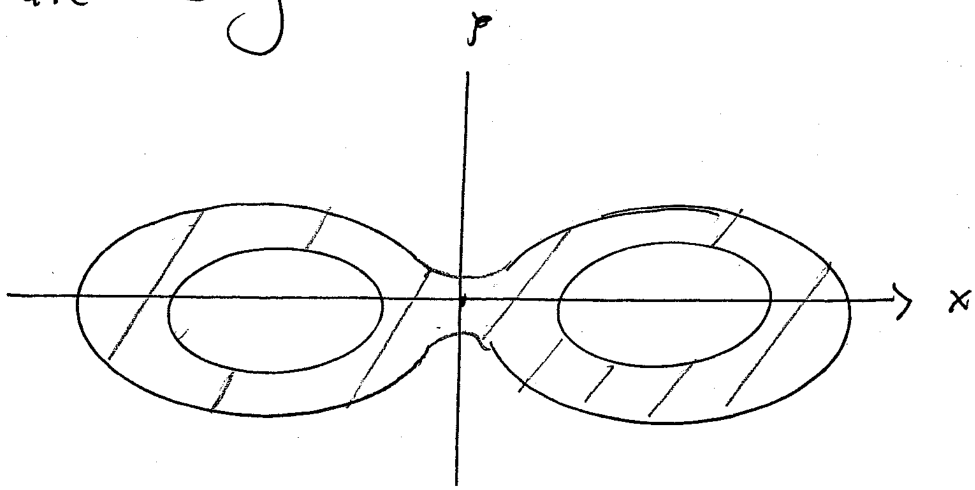
I'll conclude by saying a few words about Birkhoff canonical forms at local maxima of V. Let V be as in the figure below:



Then for $-\varepsilon < \lambda < \varepsilon$, $H(x, p) < \lambda$

looks like a slightly thickened

figure eight



The Birkhoff canonical form

in this case is

$$H_{BC} = H_{BC}(x\xi)$$

Hence for $A(\lambda)$ we get a term

coming from the graph of the

function

$$\xi = \frac{a}{x}, \qquad a = H_{BC}^{-1}(\lambda)$$

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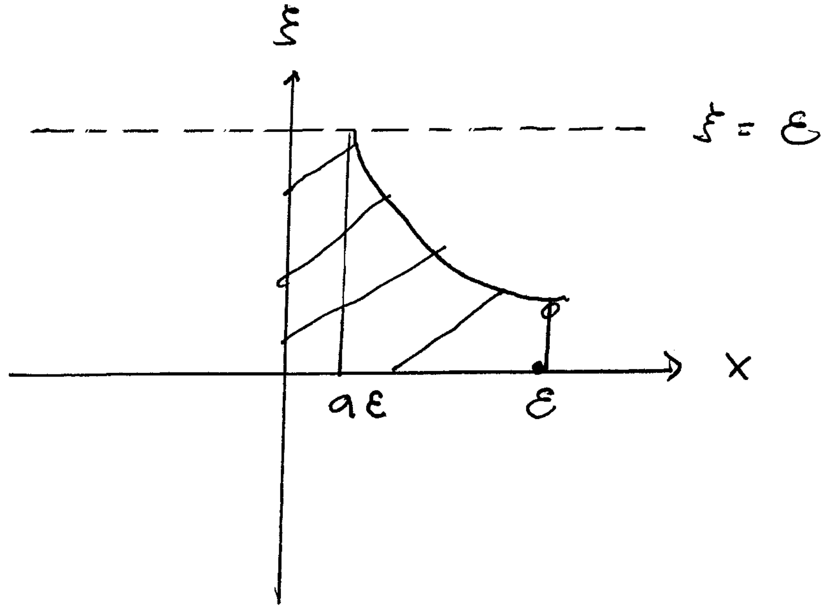

$$A = 4\left(a\varepsilon^2 + \int_{a\varepsilon}^{\varepsilon} \frac{a}{x}\,dx\right)$$

$$= 4\left(a\varepsilon^2 - a\log a\right)$$

plus a C^∞ term coming from the contribution to the area of the region outside the box $|x|, |\xi| \le \varepsilon$

Thus

$$A(\lambda) = -4\, H_{BC}^{-1}(\lambda) \operatorname{Log} H_{BC}^{-1}(\lambda) + F(\lambda)$$

where F is in $C^{\infty}(-\varepsilon, \varepsilon)$

As before the Weyl asymptotics determine the Taylor series expansion of $H_{BC}^{-1}(\lambda)$ at $\lambda = 0$. !

A quick summary of what we've proved today

1. Spectral data determine Birkhoff canonical forms

2. Modulo parity assumptions Birkhoff canonical forms determine Taylor expansions of V at critical points

Questions

1. Are these parity assumptions neccessary ?

2. Are analogues of these results true in dimension $D > 1$?