Semi-classical Birkhoff

Canonical Forms

The n-D Schroedinger operator:

$$S_h = -h^2 \Delta_{\mathbb{R}^n} + V$$

We will assume

1. $V \in C^\infty(\mathbb{R}^n)$

2. $V^{-1}((-\infty, c])$ compact

for all c.

Theorem S_h has discrete spectrum:

$$\lambda_1(h) \le \lambda_2(h) \le \cdots$$

with $\lambda_i(h) \to +\infty$ as $i \to \infty$

The inverse problem: To what extent does this spectral data determine V?

③

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be the Hamiltonian

$$H(x, \xi) = \xi^2 + V(x)$$

and

$$\dot{x}_i = \frac{\partial H}{\partial \xi_i}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial x_i}$$

the classical mechanical system with energy function H

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According to the Bohr correspondence principle, classical mechanics is the $\hbar \to 0$ limit of quantum mechanics

Semi-classical analysis : a systematic exploration of the implications of the BCP.

Some examples of semi-classical results whose implications we'll investigate in this talk.

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<u>Example 1</u> (from last time,

The Weyl law:

$\cdot \lambda)$

$$\# \{ \lambda_i(h) < \lambda \} \sim \left(\frac{1}{2\pi h}\right)^n \mathrm{vol}(H$$

Example 2 The Gutzwiller trace formula. Let

$$v_H = \sum_{i=1}^{n} \frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}$$

Suppose that for all $p \in H^{-1}(0)$

$$dH_p \neq 0$$

or alternatively

$$v_H(p) \neq 0$$

⑧

Let $\gamma(t) = \exp t\, v_H(p)$,

$p \in H^{-1}(0)$ be a periodic

trajectory of v_H of period T

i.e

$$\exp T v_H(p) = p$$

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The linear Poincaré map:

The map, $d(\exp T v_H)_p : T_p \mathbb{R}^{2n} \to T_p \mathbb{R}^{2n}$,

induces on the space

$$\text{Kernel } dH_p \big/ \{ c\, v_H(p),\ c \in \mathbb{R} \}$$

a linear map, P_γ, the

Poincaré map

<u>Definition</u> γ is non-degenerate if $\det(\underline{I} - P_\gamma) \neq 0$

Suppose γ is the only periodic trajectory of period T on the energy surface $H=0$ and is non-degenerate

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<u>Theorem</u> For $\rho \in C_0^\infty(-\varepsilon, \varepsilon)$ and $f \in C_0^\infty(-\varepsilon + T, \varepsilon + T)$ the trace of

$$\int f(t) \left(\exp \frac{u t S_h}{h}\right) \rho(S_h)\, dt$$

has an asymptotic expansion

$$(1) \qquad \exp\left(\frac{i \int_\gamma \sum \xi_i \, dx_i}{h}\right) \sum_{k=0}^{\infty} a_k h^k$$

Example 3 Density of states

Let $q \in \mathbb{R}^{2n}$ be a non-degenerate critical point of H with $H(q) = 0$

Assume: q is the only critical point of H on the energy surface, $H = 0$.

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$$v_H(q) = 0 \implies \exp t v_H(q) = q$$

<u>Definition</u> q is non-degenerate

if, for $0 < t < \varepsilon$,

$$\det\left(I - (d \exp t v_H)_q\right) \neq 0$$

Assume q is non-degenerate

<u>Theorem</u> For $0 < t < \varepsilon$ and

$\rho \in C_0^\infty(-\varepsilon, \varepsilon)$

$$(2) \qquad \operatorname{trace}\left(\frac{\exp ut S_h}{h}\right)\rho(S_h) \sim \sum_{i=0}^{\infty} a_i(t)\, h^i$$

Inverse results How do we access the information encoded in the series

$$\exp \frac{i \int_\gamma \sum \xi_i \, dx_i}{h} \quad \sum a_k h^k$$

and

$$\sum a_k(t) \, h^k \quad ?$$

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Classical and quantum

Birkhoff canonical forms

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1. Classical Birkhoff canonical forms at a critical point q of H

For simplicity let q be a non-degenerate minimum of H and let $H(q) = 0$

The map $\left(d \exp \tau v_H\right)_q : T_q \mathbb{R}^{2n} \to T_q \mathbb{R}^{2n}$

preserves $\omega_p = \left(\sum dx_i \wedge d\xi_i\right)_p$ and

$d^2 H_q$.

Hence

$$\left(d \exp t v_H\right)_q \sim \begin{bmatrix} e^{it\theta_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & e^{it\theta_n} \end{bmatrix}$$

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The Birkhoff "non-resonance" condition: The numbers, $\theta_1, \ldots, \theta_n$ are linearly independent over the rational numbers

Assume this.

<u>Theorem</u> There is a local symplectomorphism

$$\phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, q)$$

such that

$$\phi^* H = H_{CB}(p_1, \dots, p_n) + R$$

where $p_i = \frac{x_i^2 + \xi_i^2}{2}$ and R vanishes to infinite order at 0. Moreover

$$H_{CB} = \sum \theta_i p_i + O(p^2)$$

2. <u>Classical</u> <u>Birkhoff</u> <u>canonical</u> <u>forms</u> <u>at</u> <u>periodic</u> <u>trajectories</u>, γ, <u>of</u> v_H

Let γ be a non-degenerate periodic trajectory of v_H on the energy surface $H = 0$ and let P_γ be the Poincaré map associated to γ

<u>The</u> <u>elliptic</u> <u>case</u>: $P_\gamma \in \mathcal{U}(n-1)$, i.e.

$$P_\gamma \sim \begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_{n-1}} \end{bmatrix}$$

We'll restrict to this case and also normalize H so that γ has period 2π

The Birkhoff non-resonance condition :

$\theta_1, \ldots, \theta_{n-1}, \pi$ are linearly independent over the rational numbers

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Let (e^{it}, τ) be Darboux coordinates on $T^*S^1 = S^1 \times \mathbb{R}$ and let $\gamma_0 \subseteq \mathbb{R}^{2n-2} \times T^*S^1$ be the circle $x = \xi = \tau = 0$. Let's assume the non-resonance condition above holds.

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<u>Theorem</u> There is a local symplecto-morphism

$$\varphi : (\mathbb{R}^{2n-2} \times T^*S^1, \gamma_0) \to (\mathbb{R}^{2n}, \gamma)$$

such that $\varphi^* H = H_{CB}(P_1, \ldots, P_{n-1}, \tau) + R$ where $P_i = (x_i^2 + \xi_i^2)/2$ and R vanishes to infinite order on γ_B. Moreover

$$H_{CB}(p, \tau) = \tau + \frac{1}{2\pi} \sum_{i=1}^{n-1} \theta_i P_i + O(p^2).$$

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<u>Remark</u> These theorems in their modern incarnation are due to Shlomo (I learned about them in a graduate course on celestial mechanics that he taught at Harvard in the spring of 1961.)

To access the data encoded in the Gutzwiller formula and density of states we'll need quantum versions of these two results.

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The quantum version of

the Birkhoff canonical form

at a critical point q of H:

Let $\rho \in C_0^\infty(-\varepsilon, \varepsilon)$ be

identically one on $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$.

<u>Theorem</u> The operator,

$S_h \, \rho(S_h)$ is unitarily equivalent

to a Weyl operator on $L^2(\mathbb{R}^n)$

of the form

$$\tilde{H}_{QB}\left(P_1, \dots, P_n, h\right) + \tilde{R} + O(h^\infty)$$

where

a. the Weyl symbol of $\tilde{R}$

vanishes to infinite order on $p = 0$

b. the Weyl symbol of $\tilde{H}_{QB}$

is $H_{CB} + O(h^2)$

c. $P_i = -h^2 \left(\frac{\partial}{\partial x_i} \right)^2 + x_i^2$

The quantum version of the Birkhoff canonical form theorem at a periodic trajectory γ of v_H

<u>Theorem</u> The operator $S_h \, \rho(S_h)$ is unitarily equivalent to a Weyl operator on $L^2(\mathbb{R}^{n-1} \times S^1)$ of the form

$$\tilde{H}_{QB}(P_1, \dots, P_{n-1}, \mathcal{T}) + \tilde{R} + O(h^\infty)$$

where

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a. the Weyl symbol of $\tilde{R}$ vanishes to infinite order on $p = \tau = 0$

b. the Weyl symbol of $\tilde{H}_{QB}$ is of the form $H_{CB} + O(h^2)$

c. $P_i = -h^2\left(\frac{\partial}{\partial x_i}\right)^2 + x_i^2$ and $\tau = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$

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Inverse results

For simplicity I'll focus on the density of states formula

<u>Key observation</u> 1 In computing the trace of $\exp \frac{i t S_h}{h} \rho(S_h)$ one can replace S_h by $\tilde{H}_{QB}$

<u>Key observation</u> 2 The eigenvalues of P_i are $k_i + \frac{1}{2}$, $k_i = 1, 2, \ldots$

so

$$\text{trace } \exp i \frac{t}{\hbar} \tilde{H}_{\phi B}(P_1, P_2, \ldots, P_n, \hbar)$$

$$= \sum_{k} \exp it \tilde{H}_{\phi B}\left(k + \frac{\mathbb{1}}{2}\right)$$

where $\mathbb{1} = (1, 1, \ldots, 1)$

Key observation 3 (The Zelditch trick)

Let $F = \tilde{H}_{QB}(p_1, \ldots, p_n, h) - \sum \theta_k p_k$

Convert F into a differential operator by making the substitution

$$p_k \longrightarrow \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_k}$$

Then the infinite sum on the next-to-last slide can be written more compactly as

$$\exp\frac{it}{h}\, F\left(\frac{h}{t}\,\frac{1}{i}\,\frac{\partial}{\partial x}, h\right) \prod_{j=1}^{n} \frac{e^{\frac{it}{2}x_j}}{1-e^{itx_j}}\Bigg|_{\theta=x}$$

$$+\ O(h^{\infty})$$

Key observation 4

Kronecker's theorem: If $\theta_1, \dots, \theta_n$ are linearly independent over the rationals then for any n-tuple $(x_1, \dots x_n) \in \mathbb{R}^n$ there exists a sequence, $t_K \in \mathbb{R}$, such that

$$e^{i t_K \theta_r} \longrightarrow e^{i x_r}, \qquad r = 1, \dots n$$

as $K \to \infty$

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<u>Key observation</u> 5 By judicious use of Kronecker's theorem one can replace the θ_r's in Zeldich's formula by arbitrary x_r's i.e view this formula as a formal power series identity in the x_r's !

In other words one can completely recapture $\tilde{H}_{QB}$ from spectral data !

Attributional comments

1. Results of the type above for classical elliptic pseudo-differential operators were obtained by Zelditch and myself in the mid nineties

2. The <u>semi-classical</u> versions of these results were obtained by Iantchenko - Sjöstrand - Zworski about five years ago.

3. However, the idea of using Kronecker to exploit the Birkhoff non-resonance condition goes back to work of Stark and Fried in the 70's.

I'll conclude by coming back to the two questions I posed at the end of yesterday's lecture:

1. For the 1-D Schroedinger operator one can show that the results I discussed last time are true without the annoying parity restriction, $V(x) = V(-x)$, thus improving (slightly) on what Abel was able to prove 200 years ago.

Remark This is some (very recent) joint work with Yves Colin de Verdiere

2. These results can also be generalized to n-dimensions; however one still appears to need annoying parity restrictions i.e

$$(*) \qquad V(x_1, \dots, x_n) = V(\pm x_1, \dots, \pm x_n)$$

Remark: This is joint work with Alejandro Uribe

3. However maybe not <u>all</u> these conditions. For $n=2$ Yves and I can, for instance, replace (*) by

$$(**) \quad V(x_1, x_2) = V(x_1, -x_2)$$

and

$$(***) \quad V(x_1, 0) = V(-x_1, 0)$$