LECTURE 10: FREE ENTROPY AND OPERATOR ALGEBRAS

Let $(\mathcal{A}, \tau)$ be a tracial W^*-probability space and $a_1, \dots, a_n$ selfadjoint elements in $\mathcal{A}$. Recall that by definition the joint distribution of the noncommutative random variables $a_1, \dots, a_n$ is the collection of all mixed moments

$$\mathrm{dist}(a_1, \dots, a_n) = \{\tau(a_{i(1)}a_{i(2)} \dots a_{i(k)}) : k \in \mathbb{N}, i(1), \dots, i(k) \in \{1, \dots, n\}\}.$$

In this lecture we want to examine the probability that the distribution $\mathrm{dist}(a_1, \dots, a_n)$ of $(a_1, \dots, a_n)$ occurs in Voiculescu's multivariable generalization of Wigner's semicircle law.

Let $A_1, \dots, A_n$ be independent random matrices of class $\mathrm{SGRM}(N, \frac{1}{N})$. That is, $A_1, \dots, A_n$ are chosen independently at random from the sample space $M_N(\mathbb{C})_{sa}$ of $N \times N$ selfadjoint matrices over $\mathbb{C}$, equipped with Gaussian probability measure having density proportional to

$$e^{-\frac{N}{2}\,\mathrm{Tr}(M^2)}$$

with respect to Lebesgue measure on $M_N(\mathbb{C})_{sa}$. We know that as $N \to \infty$ we have almost sure convergence

$$(A_1, \dots, A_n) \to (s_1, \dots, s_n)$$

with respect to the normalized trace, where $(s_1, \dots, s_n)$ is a free semicircular family. Large deviations from this limit should be given by

$$P_N((A_1, \dots, A_n) : \mathrm{dist}(A_1, \dots, A_n) \approx \mathrm{dist}(a_1, \dots, a_n)) \sim e^{-N^2 I(a_1, \dots, a_n)},$$

where $I(a_1, \dots, a_n)$ is the "free entropy" of $a_1, \dots, a_n$. The problem is that this has to be made more precise and that, in particular, there is no analytical formula to calculate this quantity.

We use the above as motivation to define free entropy as follows. Given a tracial W^*-probability space $(\mathcal{A}, \tau)$ and an n-tuple $(a_1, \dots, a_n)$ of selfadjoint elements in $\mathcal{A}$, put

$$\begin{aligned}\Gamma(a_1, \dots, a_n; N, r, \epsilon) :=& \\ &\{(A_1, \dots, A_n) \in M_N(\mathbb{C})^n_{sa} : |\,\mathrm{tr}(A_{i_1} \dots A_{i_k}) - \tau(a_{i_1} \dots a_{i_k})| \le \epsilon \\ &\qquad \text{for all } 1 \le i_1, \dots, i_k \le n, 1 \le k \le r\}\end{aligned}$$

In words, $\Gamma(a_1, \dots, a_n; N, r, \epsilon)$ is the set of all n-tuples of $N \times N$ selfadjoint matrices which approximate the mixed moments of the selfadjoint elements $a_1, \dots, a_n$ of length at most r to within ϵ.

Let Λ denote Lebesgue measure on $M_N(\mathbb{C})^n_{sa}$. Define

$$\chi(a_1, \ldots, a_n; r, \epsilon) := \limsup_N \frac{1}{N^2} \log \Lambda(\Gamma(a_1, \ldots, a_n; N, r, \epsilon) + \frac{n}{2} \log(N),$$

and

$$\chi(a_1, \ldots, a_n) := \lim_{\substack{r\to\infty \\ \epsilon\to 0}} \chi(a_1, \ldots, a_n; r, \epsilon).$$

The function χ has the following properties:

(1) For $n = 1$, we have the explicit formula

$$\chi(a) = \int\int \log|x - y| d\mu_a(x) d\mu_a(y) + C.$$

For $n \geq 2$, it is an open problem to find a formula of this sort.

(2) χ is subadditive:

$$\chi(a_1, \ldots, a_n) \leq \chi(a_1) + \cdots + \chi(a_n).$$

This is an easy consequence of the fact that

$$\Gamma(a_1, \ldots, a_n; N, r, \epsilon) \subset \prod_{i=1}^{n} \Gamma(a_i; N, r, \epsilon).$$

Thus in particular $\chi(a_1, \ldots, a_n) \in [-\infty, \infty)$.

(3) Upper semicontinuity: if

$$(a_1^{(m)}, \ldots, a_n^{(m)}) \to (a_1, \ldots, a_n)$$

then

$$\chi(a_1, \ldots, a_n) \geq \limsup_{n\to\infty} \chi(a_1^{(m)}, \ldots, a_a^{(n)}).$$

This is because if, for arbitrary words of length k with $1 \leq k \leq r$, we have

$$|\tau(a_{i_1}^{(m)} \ldots a_{i_k}^{(m)}) - \tau(a_{i_1} \ldots a_{i_k})| < \frac{\epsilon}{2}$$

for sufficiently large m, then

$$\Gamma(a_1^{(m)}, \ldots, a_n^{(m)}; N, r, \frac{\epsilon}{2}) \subset \Gamma(a_1, \ldots, a_n; N, r, \epsilon).$$

(4) If $a_1, \ldots, a_n$ are free, then

$$\chi(a_1, \ldots, a_n) = \chi(a_1) + \ldots \chi(a_n).$$

(5) $\chi(a_1, \ldots, a_n)$ under the constraint $\sum \tau(a_i^2) = n$ has a unique maximum when $a_1, \ldots, a_n$ is a free semicircular family with $\tau(a_i^2) = 1$.

(6) Consider $y_j = F_j(x_1, \dots, x_n)$, for some "convergent" non-commutative power series F_j, such that the mapping $(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$ can be inverted by some other power series. Then

$$\chi(y_1, \dots, y_n) = \chi(x_1, \dots, x_n) + n \log(|\det|\mathcal{J}(x_1, \dots, x_n),$$

where $\mathcal{J}$ is a non-commutative Jacobian and det is the Kadison-Fuglede determinant.

1. Applications to operator algebras

One hopes that $\chi(x_1, \dots, x_n)$ enodes information about the von Neumann algebra $vN(x_1, \dots, x_n)$.

Define the "free entropy dimension" of the n-tuple $x_1, \dots, x_n$ by

$$\delta(x_1, \dots, x_n) = n + \lim_{\epsilon \searrow 0} \frac{\chi(x_1 + \epsilon s_1, \dots, x_n + \epsilon s_n)}{|\log \epsilon|},$$

where $s_1, \dots, s_n$ is a free semicircular family free from $\{x_1, \dots, x_n\}$.

The entropy dimension problem is to establish the validity (or falsehood) of the implication

$$vN(x_1, \dots, x_n) = vN(y_1, \dots, y_n) \implies \delta(x_1, \dots, x_n) = \delta(y_1, \dots, y_n).$$

In the case of free group factors $L(\mathbb{F}_n) = vN(s_1, \dots, s_n)$ we have

$$\chi(s_1, \dots, s_n) > -n \text{ and } \delta(s_1, \dots, s_n) = n.$$

Let P be some property that a vN-algebra $\mathcal{A}$ may or may not have. Assume that we can verify the implication

$$\mathcal{A} \text{ has } P \implies \chi(x_1, \dots, x_n) = -\infty$$

for any generating set $vN(x_1, \dots, x_n) = \mathcal{A}$. Then a vN-algebra for which we have at least one generating set with finite free entropy cannot have this property P.

This idea is the basis of the proof of the following theorem.

Theorem: Let $\mathcal{A}$ be a II_1-factor generated by selfadjoint operators $x_1, \dots, x_d$. Assume that $\chi(x_1, \dots, x_d) > -\infty$. Then

(1) $\mathcal{A}$ does not have property Γ (Voiculescu).
(2) $\mathcal{A}$ does not have a Cartan subalgebra (Voiculescu). Recall that a Cartan subalgebra of $\mathcal{A}$ is a maximal abelian vN-subalgebra $\mathcal{B}$ such that $\{u \in \mathcal{A} : u \text{ unitary}, u\mathcal{B}u^* = \mathcal{B}\}$ generates $\mathcal{A}$. This shows that $\mathcal{A}$ cannot be obtained from ergodic measurable relations.
(3) $\mathcal{A}$ is prime, i.e. $\mathcal{A}$ cannot be decomposed as $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ for II_1-factors $\mathcal{A}_1, \mathcal{A}_2$ (Liming Ge).

Corollary: All this applies in the case of the free group factor $L(\mathbb{F}_n)$.

We will prove part (1) of the above theorem, although the absence of property Γ for $L(\mathbb{F}_n)$ is an old result of Murray and von Neumann which can be proved more directly without using free entropy.

Definition: A bounded sequence $(t_k)_{k\geq 0}$ in $(\mathcal{A},\tau)$ is *central* if

$$\lim_{k\to\infty} \|xt_k - t_k x\|_2 = 0$$

for all $x \in \mathcal{A}$. If

$$\lim_{k\to\infty} \|t_k - \tau(t_k)\cdot 1\|_2 = 0,$$

then $(t_k)_k$ is said to be *trivial.* $(\mathcal{A},\tau)$ has property Γ if there exists a non-trivial central sequence in $\mathcal{A}$.

2. Sketch of Proof

We now give an outline of Part (1) of the above Theorem, namely that when selfadjoint operators $x_1,\dots,x_d$ have the property that $\chi(x_1,\dots,x_d) > -\infty$, then the von Neumann algebra they generate cannot have property Γ.

So let $\mathcal{A} = vN(x_1,\dots,x_d)$ have property Γ; we must prove that this implies $\chi(x_1,\dots,x_d) = -\infty$. Let $(t_k)_k$ be a non-trivial central sequence in $\mathcal{A}$.

By applying functional calculus to this sequence, we may replace the t_k's with a non-trivial central sequence of orthogonal projections $(p_k)_k$, and assume the existence of a real number θ in the open interval $(0,1/2)$ such that

$$\theta < \tau(p_k) < 1-\theta \text{ for all } k$$

and

$$\lim_{k\to\infty} \|[x,p_k]\|_2 = 0 \text{ for all } x\in\mathcal{A}.$$

We then prove the following key lemma:

Lemma: Let $(\mathcal{A},\tau)$ be a W^*-probability space generated by selfadjoint elements $x_1,\dots,x_d$ satisfying $\tau(x_i^2) \leq 1$. Let $0<\theta<\frac{1}{2}$ be a constant and $p\in\mathcal{A}$ a projection such that $\theta<\tau(p)<1-\theta$ and $\|[p,x_i]\|_2 < \omega$. Then there exist positive constants C_1, C_2 depending only on d and θ such that

$$\chi(x_1,\dots,x_d) \leq C_1 + C_2\log\omega.$$

Assuming this is proved, choose $p = p_k$. If $k \to \infty$, we can take $\omega \to 0$. Thus we get

$$\chi(x_1, \ldots, x_d) \leq C_1 + C_2 \log \omega$$

for all $\omega > 0$, implying

$$\chi(x_1, \ldots, x_d) = -\infty.$$

It remains to prove the lemma. Take

$$(A_1, \ldots, A_d) \in \Gamma(x_1, \ldots, x_d; N, \epsilon, k)$$

for N, k sufficiently large and ϵ sufficiently small. As p can be approximated by polynomials in $x_1, \ldots, x_d$ and by deforming things again a bit by functional calculus, we find a projection matrix $Q \in M_N(\mathbb{C})$ whose range is a subspace of dimension $q = \lfloor N\tau(p) \rfloor$ and such that

$$\|[A_i, Q]\|_2 < 2\omega.$$

This Q is of the form

$$Q = U \begin{pmatrix} I_q & 0 \\ 0 & 0_{N-q} \end{pmatrix} U^*$$

for some $U \in \mathcal{U}(N)/\mathcal{U}(q) \times \mathcal{U}(N-q)$. Write

$$UA_i^*U = \begin{pmatrix} B_i & C_i^* \\ C_i & D_i \end{pmatrix}$$

Then

$$\|[A_i, Q]\|_2 \leq 2\omega \implies \| \begin{pmatrix} B_i & C_i^* \\ C_i & D_i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \|_2 < 2\omega,$$

and

$$\| \begin{pmatrix} 0 & -C_i \\ C_i^* & 0 \end{pmatrix} \|_2 = \sqrt{\frac{2}{N} \operatorname{Tr}(C_i C_i^*)} \implies \operatorname{Tr}(C_i C_I^*) < \frac{N}{2}(2\omega)^2 = 2N\omega^2.$$

Now

$$\tau(x_i^2) \leq 1 \implies \operatorname{tr}(A_i^2) \leq 1 + \epsilon \implies \operatorname{Tr}(A_i^2) \leq (1+\epsilon)N \leq 2N$$

for $\epsilon \leq 1$.

Denote by $B_p(R)$ the ball of radius R in $\mathbb{R}^p$ centred at the origin. The Lebesgue measure of $B_p(R)$ is given in terms of the Gamma function as

$$\Lambda(B_p(R)) = \frac{R^p \pi^{p/2}}{\Gamma(1 + \frac{p}{2})}.$$

Now cover $\Gamma(x_1, \ldots, x_d; N, \epsilon, k)$ by a union of products of balls:

$$\Gamma(x_1, \ldots, x_d; N, \epsilon, k) \subseteq \bigcup_{U \in \mathcal{U}(N)/\mathcal{U}(q)\times\mathcal{U}(N-q)} [U^* B_{q^2}(\sqrt{2N}) \times B_{2q(N-q)}(\omega\sqrt{2N}) \times B_{(N-q)^2}(\sqrt{2N}) U]^d.$$

This does not give directly an estimate for the volume of our set Γ, as we have here a covering by infinitely many sets. However, we can reduce this to a finite cover by approximating the appearing U's from elements from a finite δ-net.

By a result of Szarek, for any $\delta > 0$ there exists a δ-net $(U_s)_{s\in S}$ in the Grassmannian $\mathcal{U}(N)/\mathcal{U}(q) \times \mathcal{U}(N-q)$ with

$$|S| \leq (C\delta^{-1})^{N^2-q^2-(N-q)^2}$$

with C a universal constant.

For $(A_1, \ldots, A_d), Q, U$ as above, we have: there exists $s \in S$ such that $\|U - U_s\| \leq \delta$ implies

$$\|[U_s^* A_i U_s, U^* Q U]\|_2 \leq 2\omega + 8\delta.$$

Thus covering $\Gamma(x_1, \ldots, x_d; N, \epsilon, k)$ with the δ-net we have

$$\Gamma(x_1, \ldots, x_d; N, \epsilon, k) \subseteq \bigcup_{s\in S} [U_s^* B_{q^2}(\sqrt{2N}) \times B_{2q(N-q)}(\omega\sqrt{2N}) \times B_{(N-q)^2}(\sqrt{2N}) U_s]^d,$$

and hence

$$\Lambda(\Gamma(x_1, \ldots, x_d; N, \epsilon, k)) \leq$$
$$(C\delta^{-1})^{N^2-q^2-(N-q)^2} \Lambda(U_s^* B_{q^2}(\sqrt{2N}))^d \Lambda(B_{2q(N-q)}(\omega\sqrt{2N}))^d \Lambda(B_{(N-q)^2}(\sqrt{2N}) U_s)^d,$$

which simplifies to the bound

$$(C\delta^{-1})^{2q(N-q)} \frac{(2N)^{(N-q)^2/2} \pi^{N^2/2}}{\Gamma(1+q^2/2)^d \Gamma(1+q(N-q))^d \Gamma(1+(N-q)^2/2)^d}.$$

Thus

$$\frac{1}{N^2} \log \Lambda(\Gamma(x_1, \ldots, x_d; N, \epsilon, k)) + \frac{d}{2} \log N \leq$$
$$C_1 + \frac{2q(N-q)}{N^2}(\log \delta^{-1} + d \log(\omega + 4\delta)) \leq \tilde{C} \log \omega$$

for $d \geq 2$, where the new constant $\tilde{C} \geq 0$ is gotten by taking $\delta = \omega:$

$$\tilde{C} = -\log \omega + d \log \omega + d \log 4.$$