The Fields Institute
Free Probability and Random Matrices

Problem Set 5, Due December 13, 2007

Do two of the three questions

1) (a) Let $i_1, i_2, i_3, \ldots, i_n \in \{1, 2\}$, and for n even $NC_2^{(i)}(n)$ be the non-crossing pairings, π, of $[n]$ such that for each pair (r, s) of π we have $i_r = i_s$. Let s_1 and s_2 be free semi-circular elements in $(\mathcal{A}, \phi)$ with $\phi(s_1^2) = \phi(s_2^2) = 1$. Show that for n even

$$\phi(s_{i_1} s_{i_2} \cdots s_{i_n}) = |NC_2^{(i)}(n)|;$$

and for n odd

$$\phi(s_{i_1} s_{i_2} \cdots s_{i_n}) = 0$$

(b) Suppose that s_1 and s_2 are elements of $(\mathcal{A}, \phi)$ with $\phi(s_1^2) = \phi(s_2^2) = 1$ and $i_1, i_2, i_3, \ldots, i_n \in \{1, 2\}$. Suppose we have for n even

$$\phi(s_{i_1} s_{i_2} \cdots s_{i_n}) = |NC_2^{(i)}(n)|$$

and for n odd $\phi(s_{i_1} s_{i_2} \cdots s_{i_n}) = 0$. Show that s_1 and s_2 are free.

(c) Suppose that $s_1, s_2, c \in \mathcal{A}$ with s_1 and s_2 semi-circular and c circular. Let $x = \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \in M_2(\mathcal{A})$. Define a state, ψ, on $M_2(\mathcal{A})$ by $\psi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2}\phi(a_{11} + a_{22})$. Show that x is semi-circular by calculating $\psi(x^k)$ for arbitrary k.

(d) Let $s_1, s_2, s_3, s_4, c_1, c_2 \in \mathcal{A}$ be such that

- s_1, s_2, s_3, s_4 are semi-circular with $\phi(s_i^2) = 1$ for $i = 1, 2, 3, 4$;
- c_1, c_2 are circular with $\phi(c_i^* c_i) = 1$ for $i = 1, 2$;
- $s_1, s_2, s_3, s_4, c_1, c_2$ are *-free.

Let $x_1 = \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}$. Show by using part (b) above that x_1 and x_2 are free and semi-circular.

2) An alternative approach to free entropy relies on the free Fisher information, denoted by Φ^*. This question develops some of its basic properties.

If $(\mathcal{A}, \varphi)$ is a tracial W^*-probability space and $X \in \mathcal{A}$ is selfadjoint, then an element $J \in L^2(X, \varphi)$ is called a conjugate variable for X if we have for all $n = 0, 1, 2, \cdots$ that

$$\varphi(JX^n) = \sum_{k=0}^{n-1} \varphi(X^k)\varphi(X^{n-k-1}) \qquad (*)$$

(For $n = 0$ read this as $\varphi(J) = 0$.)
($L^2(X, \varphi)$ is here the closure of polynomials in X under the norm $\|a\|_2^2 := \varphi(aa^*)$.)

We put (after having proved part (a) below)

$$\Phi^*(X) := \begin{cases} \varphi(J^2), & \text{if a conjugate variable } J \text{ exists} \\ +\infty, & \text{otherwise} \end{cases}$$

a) Show that there is at most one J with these properties and that it must be selfadjoint!

b) Prove the free Fisher-Rao inequality:

$$\Phi^*(X) \geq \frac{1}{\varphi(X^2)}.$$

c) Reformulate the condition $(*)$ in terms of free cumulants of the form $\kappa_n(J, X, X, \dots, X)$ (one J as argument and $n-1$ X as argument)!

d) Find the conjugate variable and free Fisher information of a semicircular element.

e) Show that if one finds an element $\xi \in L^2(\mathcal{A}, \varphi)$ which satisfies $(*)$ then the conjugate variable J of X exists and is given by $J = E(\xi)$, where $E : L^2(\mathcal{A}) \to L^2(X)$ is the orthogonal projection onto $L^2(X)$.

f) Use part (e) to show: If $X = X^*$ and $Y = Y^*$ are free, then

$$\Phi^*(X + Y) \leq \Phi^*(X) + \Phi^*(Y).$$

3) This exercise concerns the Brown measure presented in lecture 9; let us briefly recall the definitions. Let M be a von Neumann factor of type II_1 with faithful normal trace τ. Given T in M, recall that the Fuglede-Kadison determinant $\Delta(T)$ of T is defined to be $\exp(\tau(\log(|T|)))$. The Brown measure μ_T of T is defined to be $\nabla_\lambda^2(\log(\Delta(T - \lambda 1)))$, where $\lambda = x + iy$, $\nabla_\lambda^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and all derivatives are in the distributional sense. It was shown in the lecture that μ_T is a Borel probability measure with support contained in the the spectrum of T.

Let $p(z)$ be a polynomial in z and ν any Borel probability measure on $\mathbb{C}$. Define a new Borel probability measure ν_p on $\mathbb{C}$ by $\nu_p(E) = \nu(p^{-1}(E))$ for any Borel subset E.

Show that $\mu_{p(T)} = (\mu_T)_p$.