Stable Traffic Equilibria: Properties and Application

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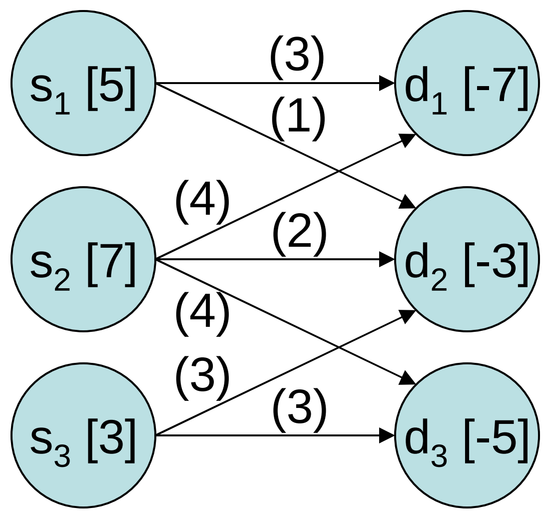
(in part based on work of Yurii Nesterov's)

TRAFFIC JAM
EXTREME!
5th Avenue
X 99
reset
pause
solve!
menu
AMORSOLO ST.
PUZZLE 3 OF 20
SCORE:
139,972
LVL SCORE:
2,071

Some history

- Transportation problem
  - formulated in 1781 by Gaspard Monge
  - major advances during WWII by Leonid Kantorovich
  - a predecessor to *linear programming*
  - in a simple form
    - given a set of balanced demands and supplies and corresponding shipping costs, find the cheapest routing, i.e., a minima

	d_1	d_2	d_3	Supply
s_1	2	3	0	5
s_2	5	0	2	7
s_3	0	0	3	3
Demand	7	3	5	



Some history

- Transportation problem
  - formulated in 1781 by Gaspard Monge
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  - in a simple form
    - given a set of balanced demands and supplies
      and corresponding shipping costs,
      find the cheapest routing, i.e., a minima
  - generalizes to network flows (single and multi-commodity), transportation networks (network equilibrium, Beckmann, McGuire and Winsten, 1956), etc.

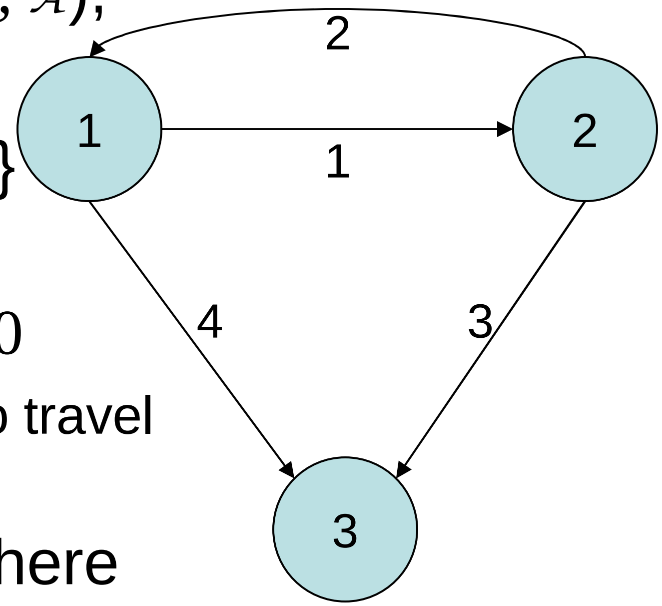
Network equilibria

- Concept originated in 1924 and is due to Frank Knight
- Intuitively clear meaning:

  “a stable state of the traffic network, i.e., a state that can sustain itself over time”

  – may be interpreted on the appropriate time-scale, e.g., a “steady” traffic pattern during the rush hour
- Formalized by John Glen Wardrop in 1952
  – **First principle:** “*The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route*”, i.e., selfish agents
  – **Second principle:** “*At equilibrium the average journey time is minimum* ”, i.e., efficient use of the whole system

Modeling formalism



Network R, given as a directed graph $(N, \mathcal{A})$,

- e.g., N={1,2,3}, $\mathcal{A}$ ={1,2,3,4}

OD-pairs $OD=\{(i_k, j_k): i_k, j_k \in N, k=1,\dots,P\}$

- e.g., OD={(1,2),(1,3),(2,1),(2,3)}

Each OD-pair generates a demand $d_k \geq 0$

- an average flow of drivers that needs to travel between i_k and j_k

Travel cost (time) $c_\alpha(f_\alpha)$, $\alpha=1,\dots,||\mathcal{A}||$, where

$f_\alpha \geq 0$ is the flow on arc α

Assume: c_α *is increasing with* f_α

For each (i_k, j_k) define set of routes A_k, e.g., $A_2 = \begin{pmatrix} & 1 \\ & \\ & 1 \\ 1 & \end{pmatrix}$

Route 2

Arc 3

where number of such routes is denoted r_k

Modeling formalism

For each (i_k, j_k) introduce a set feasible route flows

$\Delta_k = \{F_k \in \mathbb{R}_+^{r_k} : \mathbf{1}^T F_k = d_k\}$

Note, for any choice of route flows $F=(F_1, \ldots, F_k) \in \Delta$

can compute the vector of arc flows $f = AF$,

where $A=(A_1,\ldots,A_P)$,

and compute cost of the routes as $C_k(F)= A_k^T c(AF)$

Formally, static (Waldrop) equilibrium is $F \in \Delta$ such that

$$F_k^{(m)} > 0 \Rightarrow C_k^{(m)}(F) = \min_r C_k^{(r)}(F)$$

Alternatively

$$\langle C(F), \bar{F} - F \rangle \geq 0, \text{ for all } \bar{F} \in \Delta$$

$$\min_{f,F} \{\Sigma_\alpha \int_0^{f_\alpha} c_\alpha(z)\, dz : f = AF, F \in \Delta\}$$

note convex minimization problem

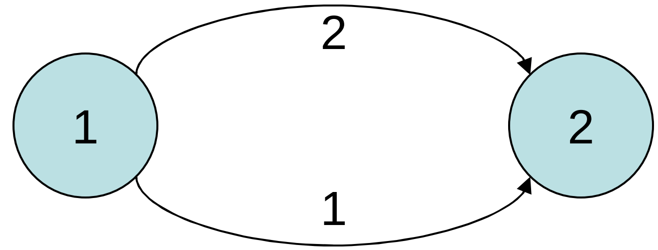
Equilibrium existence

- Wardrop
  - for any OD-pair the cost for all used paths are the same and there are no unused paths with strictly smaller cost
- User
  - for any OD-pair, no arbitrary small pack of drivers can benefit by switching to another path

Note: if cost f is continuous, these are the same

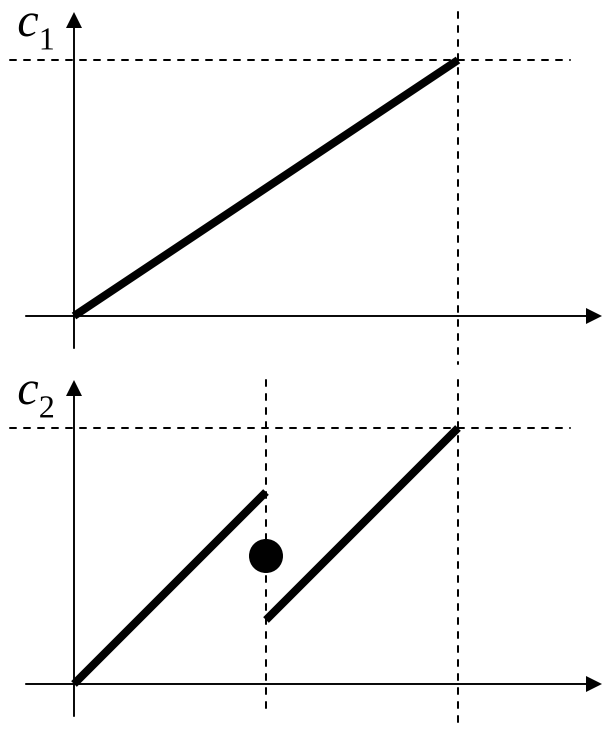
But, what if not ?

Equilibrium existence

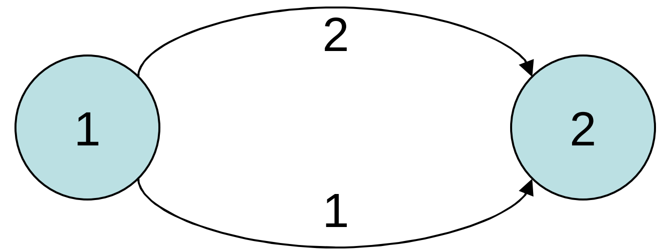


Three examples:

- Wardrop but no user
  - source and sink, unit demand, two arcs
  - $c_1(f_1) = \quad f_1,$
  - $c_2(f_2) = \begin{cases} 5/4\, f_2, & \text{if } f_2<1/2 \\ 1/2, & \text{if } f_2=1/2 \\ 5/4\, f_2 - 1/4, & \text{if } f_2>1/2 \end{cases}$
  - then $f_1 = f_1 = 1/2$ is a Wardrop equilibrium

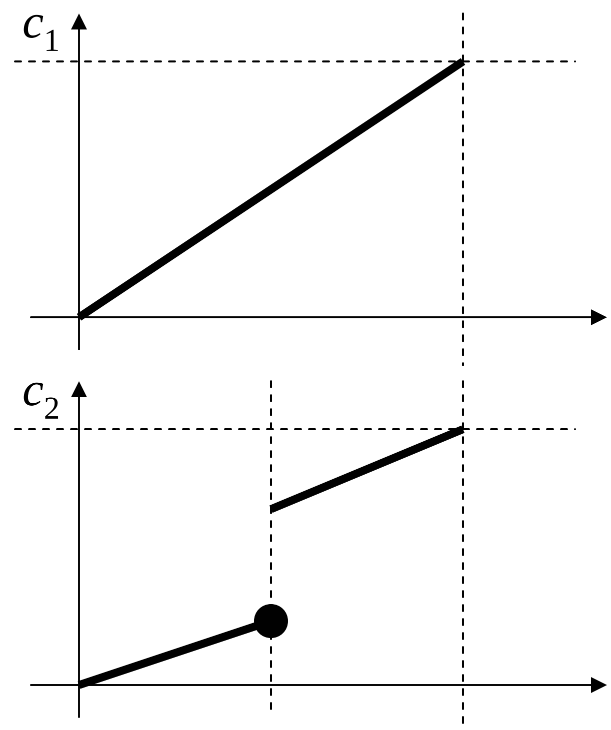


Equilibrium existence

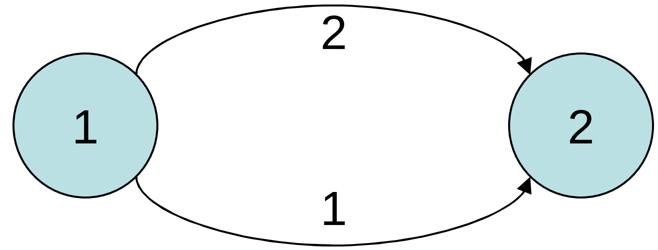


Three examples:

- user but no Wardrop
  - source and sink, unit demand, two arcs
  - $c_1(f_1) = \quad f_1,$
  - $c_2(f_2) = \quad 1/2\ f_2, \qquad \text{if } f_2 \leq 1/2$
    $\qquad\qquad\quad 1/2\ f_2 + 1/2, \qquad \text{if } f_2 > 1/2$
  - then $f_1 = f_1 = 1/2$ is a user equilibrium

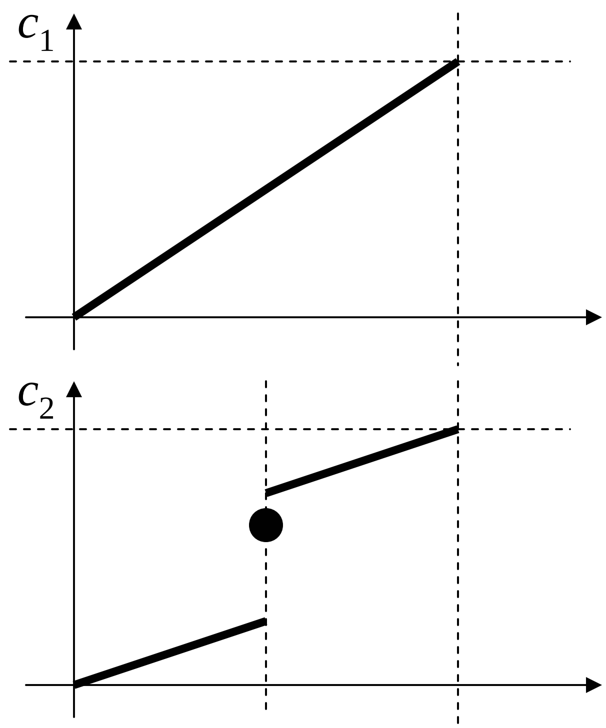


Equilibrium existence



Three examples:

- neither Wardrop nor user
  - source and sink, unit demand, two arcs
  - $c_1(f_1) = f_1$,
  - $c_2(f_2) = \begin{cases} 1/2\, f_2, & \text{if } f_2<1/2 \\ 5/8, & \text{if } f_2=1/2 \\ 1/2\, f_2 + 1/2, & \text{if } f_2>1/2 \end{cases}$
  - then no equilibrium



Braess's paradox

- Adding a shortcut can slow down everyone!

Braess's paradox

Braess's paradox

Modeling formalism

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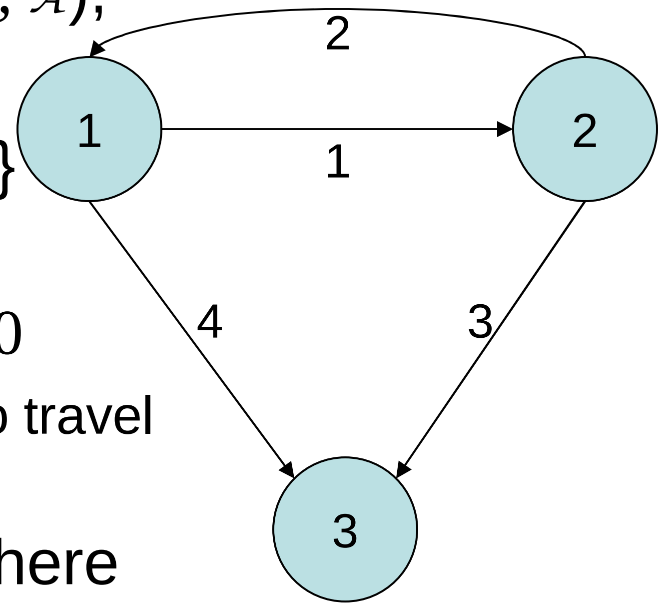

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Each OD-pair generates a demand $d_k \geq 0$

- an average flow of drivers that needs to travel between i_k and j_k

Travel cost (time) $c_\alpha(f_\alpha)$, α =1,…,||$\mathcal{A}$ ||, where

$f_\alpha \geq 0$ is the flow on arc α

Assume: c_α *is increasing with* f_α

For each (i_k , j_k) define set of routes A_k, e.g., A_2=

Route 2

	1
	1
1	

Arc 3

where number of such routes is denoted r_k

Nesterov’s approach

- Main problem with a classical model?
  - Cost (time) increases with flow
    - e.g., a free interstate gives large flow, short time, or a congested route with small flow, long time

flow = speed x density

- An alternative?
  - Queering model
    - Time(n number of travelers) = min { T, n / f }
      where T is free traffic time,
      f – capacity of the bottleneck

Modeling formalism

- Additional ingredients
  - arc flows f
  - arc travel times t
  - arc loading n (average number of drivers on the arc)

- Network state is stable if $n = t\,f$
- Analysis is in terms of network loading
  - e.g., N_k – average number of drivers between OD-pair

“Potential” function

$$\phi(t) = \sum_{k=1}^{P} N_k \ln T_k(t)$$

where $T_k(t)$ is the shortest path (w.r.t. t) between OD-pair k

Note: the function is concave in t

Equilibrium characterization

- Network state (f, t, n) is stable (Wardrop) equilibrium iff it is stable and

$$f \in \partial\phi(t)$$

  – $\partial\phi$ is superdifferential, e.g., a derivative if ϕ is differentiable

- As a corollary, f corresponds to a stable equilibrium iff $\max_t\{\phi(t) - \langle f, t \rangle\}$ is solvable

Tractability and comparison

- Sufficient conditions for equilibrium
  - $t \geq \underline{t}$ (natural lower bound on travel times)
  - $f \leq \bar{f}$ (natural upper bound on arc capacities)
- Using the potential function, can rephrase equilibrium in terms of OD-demands
- Problem allows reformulation as LP (for efficient computation)
- Consistent with queering model
- Classical Beckmann model detects congestions “too late” and, thus, systematically underestimates congestion

Extensions

- Tractable network design is possible (over arc capacities and shortest times)
  - minimize "social distance" (average travel time), as a byproduct increase congestion on most attractive arcs
  - maximize "social distance" while offloading most popular (congested) arcs
    - e.g., traffic lights or lane direction inversion

Analysis ingredients

- Graph theory
- Variation inequalities/calculus
- Convex analysis/optimization

- Selected references
  - Yurii Nesterov, *Stable Traffic Equilibria: Properties and Applications*
  - Andre de Palma, Yurii Nesterov, *Optimization Formulations and Static Equilibrium in Congested Transportation Networks*