Examples of quantum cluster algebras related to partial flag varieties

Jan E. Grabowski

Mathematical Institute, University of Oxford

Partial flag varieties

Let C be an $l \times l$ generalized Cartan matrix with columns indexed by a set I. Let $(H, \Pi, \Pi^\vee)$ be a minimal realization of C with $H \cong \mathbb{C}^{2l-\mathrm{rank}(C)}$, $\Pi = \{\alpha_i \mid i \in I\} \subset H^*$ (the fundamental roots) and $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset H$. Then we say $\mathcal{C} = (C, I, H, \Pi, \Pi^\vee)$ is a root datum associated to C.

Given a root datum $\mathcal{C}$ as above, choose a subset J of the indexing set I (i.e. a subset of nodes of the Dynkin diagram). Then we can naturally form a *sub-root datum* $\mathcal{C}' = (C', J, H', \Pi', (\Pi')^\vee)$ whose Cartan matrix C' is the submatrix of C defined by J. Set $D = I \setminus J$.

Associated to a pair $\mathcal{C}' \subseteq \mathcal{C}$ are several algebraic and geometric objects:

- If G is a complex algebraic group associated to $\mathcal{C}$, G has a (standard) parabolic subgroup P_J with opposite unipotent radical, N_D^-.
- We can form G/P_J, the *partial flag variety.*
(The case $A_0 \subseteq \mathcal{C}$ gives the complete (or full) flag variety G/B.)
- Via the Plücker embedding, we may form the corresponding $\mathbb{N}^D$-graded multi-homogeneous coordinate algebras $\mathbb{C}[N_D^-]$ and $\mathbb{C}[G/P_J]$.
- The coordinate ring $\mathbb{C}[G]$ has a quantum analogue, $\mathbb{C}_q[G]$ (also denoted $\mathcal{O}_q(G)$). From this we can define a quantization $\mathbb{C}_q[G/P_J]$ and a localisation of the latter whose degree 0 part is isomorphic to $\mathbb{C}_q[N_D^-]$.
- G has a Lie algebra $\mathfrak{g}$ with quantized enveloping algebra $U_q(\mathfrak{g})$ having a subalgebra $U_q(\mathfrak{p}_J) \stackrel{\mathrm{def}}{=} U_q(\mathfrak{g}')U_q^+(\mathfrak{g})$ analogous to P_J ($\mathfrak{g}'$ of type $\mathcal{C}'$).
- Dual to $\mathbb{C}_q[N_D^-]$ is a braided Hopf algebra $U_q(\mathfrak{n}_D^-) \cong U_q(\mathfrak{g})/U_q(\mathfrak{p}_J)$ ([1]).

Cluster algebras and their quantum analogues

Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky ([2]) in 2001, providing a framework for combinatorics associated to canonical bases of quantum groups and total positivity for semisimple algebraic groups. One way of thinking of what it means for a (necessarily) commutative algebra to possess a cluster algebra structure is that it has a particular form of presentation, with many generators but relatively simple relations.

Much work has been done in recent years on

classification: via Cartan matrices (hence rank, finite/infinite types…)

combinatorics: e.g. reformulations via quivers, geometric realisations

categorification: "cluster categories" of modules whose tilting theory encodes the cluster combinatorics

Examples include polynomial algebras (rank 0) and coordinate algebras, e.g. $\mathbb{C}[SL_2]$ (of cluster algebra type A_1) and $\mathbb{C}[SL_4/N]$, N upper unitriangular matrices (of cluster algebra type A_3).

We will focus on the so-called geometric type and describe in detail only the "no coefficients" case. We start with an *initial seed* consisting of $\underline{y}$, a tuple of generators (called a *cluster*) for the cluster algebra, and an *exchange matrix* B. More seeds are obtained via *mutation* of the initial seed. Matrix mutation μ_k is involutive and given by the rule

$$(\mu_k(B))_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & \text{otherwise} \end{cases}$$

If $(\underline{y} = (y_1, \dots, y_d), B)$ is the initial seed then the mutated seed in direction k is given by $(\mu_k(\underline{y}) = (y_1, \dots, y_k^*, \dots, y_d), \mu_k(B))$, where the new generator y_k^* is determined by the *exchange relation*

$$y_k y_k^* = \prod_{b_{ik}>0} y_i^{b_{ik}} + \prod_{b_{ik}<0} y_i^{-b_{ik}}$$

The "with coefficients" version includes additional generators present in every cluster that are never mutated but monomials in them also appear as coefficients in the exchange relations. The generators that are not coefficients will be called "mutable". We will indicate the mutable variables by boldface type.

An equivalent approach takes the quiver $\Gamma(B)$ defined by B, with arrows arising from non-zero matrix entries, and gives a more easily applied pictorial rule for mutation. Coefficients give rise to "frozen" vertices (boxed in the diagrams).

Cluster algebras associated to partial flag varieties

Work of Geiß, Leclerc and Schröer ([3]) has identified cluster algebra structures on $\mathbb{C}[G/P_J]$ and $\mathbb{C}[N_D^-]$ associated to $\mathcal{C}' \subseteq \mathcal{C}$ as above. Their approach produces the cluster algebra structure from a categorification, specifically from a subcategory of $\Lambda(\mathcal{C})$-mod, the category of modules of the preprojective algebra of type $\mathcal{C}$. The complete rigid modules in this subcategory correspond to the clusters of $\mathbb{C}[N_D^-]$ and mutation arises from certain short exact sequences.

Geiß, Leclerc and Schröer have also shown that monomials in the variables appearing in a single cluster are elements of the dual semicanonical basis of $\mathbb{C}[N_\emptyset^-]$. It is conjectured that these monomials lie in the dual *canonical* basis.

Quantum cluster algebras

Berenstein and Zelevinsky ([4]) have given a definition of a quantum cluster algebra. These algebras are now non-commutative but not so far from being commutative. Each quantum seed includes an additional piece of data, a skew-symmetric matrix L describing *quasi-commutation* relations between the variables in the cluster. (Quasi-commuting means $ab = q^{L_{ab}}ba$, also written $[a,b]_{q^{L_{ab}}} = 0$.)

There is also a mutation rule for these quasi-commutation matrices and a modified exchange relation that involves further coefficients that are powers of q derived from B and L. The natural requirement that all mutated clusters also quasi-commute leads to a compatibility condition between B and L, namely that $B^T L$ consists of two blocks, one diagonal with positive integer diagonal entries and one zero.

Importantly, Berenstein and Zelevinsky show that the exchange graph (whose vertices are the clusters and edges are mutations) remains unchanged in the quantum setting. That is, the matrix L does not influence the exchange graph. It follows that quantum cluster algebras are classified by Dynkin types in exactly the same way as the classical cluster algebras.

Previously-studied examples of quantum cluster algebras include quantum symmetric algebras (of rank 0) and quantum double Bruhat cells (given in [4]). We now give the example of $\mathbb{C}_q[SL_2]$ to illustrate the above definitions.

Example. Let $\mathcal{A} = \mathbb{C}[b, c]$ and set

$$\mathcal{A}_q(a, d) = \mathcal{A}<a, d>/<[\,a, b\,]_q, [\,a, c\,]_q, [\,d, b\,]_{q^{-1}}, [\,d, c\,]_{q^{-1}}>$$

and

$$\mathbb{C}_q[SL_2] = \mathcal{A}_q(a, b)/<ad = 1 + qbc,\ da = 1 + q^{-1}bc>.$$

This is a quantum cluster algebra of type A_1 with initial seed $((\boldsymbol{a}, b, c), B, L)$ where

$$B = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \qquad \Gamma(B) = \begin{tikzcd} 1 \arrow[r] \arrow[d] & \boxed{2} \\ \boxed{3} & \end{tikzcd}$$

$$L = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \Gamma(L) = \begin{tikzcd} 1 \arrow[r] \arrow[d] & \boxed{2} \\ \boxed{3} & \end{tikzcd}$$

Then $B^T L = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}$. The two clusters are $(\boldsymbol{a}, b, c)$ and $(\boldsymbol{d}, b, c)$. Setting $X_1 = \boldsymbol{a}$, $X_2 = b$ and $X_3 = c$, the first quantum exchange relation is calculated as

$$\begin{aligned} X_1' &= q^0 X_1^{-1} + q^{\frac{1}{2}(-l_{21} - l_{31} + l_{32})} X_1^{-1} X_2 X_3 \\ &= \boldsymbol{a}^{-1} + q\boldsymbol{a}^{-1} bc. \end{aligned}$$

That is, $X_1' = \boldsymbol{d}$ and $\boldsymbol{ad} = 1 + qbc$.

Note that forgetting about L and setting $q = 1$ we recover $\mathbb{C}[SL_2]$ as $\mathcal{A}_1(\boldsymbol{a}, \boldsymbol{d})/<\boldsymbol{ad} = 1 + bc>$ with its usual cluster algebra structure.

Examples of quantum cluster algebras

Example: complex projective space

The partial flag variety obtained from $G = G(A_n) = SL_{n+1}(\mathbb{C})$, $J = I \setminus \{n\}$ is $G/P_J = \mathbb{CP}^n$, complex projective space. The corresponding quantized coordinate ring $\mathbb{C}_q[\mathbb{CP}^n]$ is $S_q(\mathbb{C}^{n+1})$, a quantum symmetric algebra, thus of rank 0 as a quantum cluster algebra.

The unipotent radical $N^-_{\{n\}}$ is $\mathbb{C}^n$, i.e. affine space of dimension n, and its quantized coordinate ring $\mathbb{C}_q[\mathbb{C}^n]$ is $S_q(\mathbb{C}^n)$, so is also a rank 0 quantum cluster algebra. The dual to this, $U_q(\mathfrak{n}^-_{\{n\}})$, is again a quantum symmetric algebra on n variables: the Lie algebra $\mathfrak{n}^-_D$ is the n-dimensional natural $\mathfrak{sl}_n$-module V with the zero Lie bracket, having universal enveloping algebra $U(\mathfrak{n}^-_D) \cong S(V)$.

Example: the Grassmannian $\mathrm{Gr}(2,5)$

The partial flag variety

The partial flag variety obtained from $G = G(A_4) = SL_5(\mathbb{C})$, $J = I \setminus \{2\}$ is $G/P_J = \mathrm{Gr}(2,5)$, the Grassmannian of planes in $\mathbb{C}^5$. Its quantized coordinate ring $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ is the subalgebra of the quantum matrix algebra $\mathbb{C}_q[M(2,5)]$ generated by the quantum Plücker coordinates, as follows.

The quantum matrix algebra $\mathbb{C}_q[M(2,5)]$ is generated by the set $X = \{x_{ij} \mid 1 \leq i \leq 2,\ i \leq j \leq 5\}$ subject to the quantum 2×2 matrix relations on each 2×2 submatrix of

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \end{pmatrix},$$

where the quantum 2×2 matrix relations on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are

$$\begin{array}{lll} ab = qba & ac = qca & bc = cb \\ bd = qdb & cd = qdc & ad - da = (q - q^{-1})bc. \end{array}$$

Hence a presentation for $\mathbb{C}_q[M(2,5)]$ is

$$\begin{aligned} \mathbb{C}_q[M(2,5)] = \mathbb{C}{<}X{>} \ / & \\ & {<}[\,x_{1i}, x_{1j}\,]_q,\ [\,x_{2i}, x_{2j}\,]_q,\ [\,x_{1i}, x_{2i}\,]_q,\ [\,x_{2i}, x_{1j}\,]\ , \\ & \ [\,x_{1i}, x_{2j}\,]\ = (q - q^{-1})x_{1j}x_{2i} \quad \forall\ 1 \leq i < j \leq 5{>} \end{aligned}$$

The quantum Plücker coordinates that generate $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ are the 2×2 quantum minors

$$\mathcal{P}_q = \{\Delta_q^{ij} \stackrel{\mathrm{def}}{=} x_{1i}x_{2j} - qx_{1j}x_{2i} \mid 1 \leq i < j \leq 5\}.$$

As described in [5] and [6], two quantum Plücker coordinates Δ_q^{ij} and Δ_q^{kl} quasi-commute if and only if $\{i,j\}$ and $\{k,l\}$ are weakly separated. In this particular case, this means that the corresponding diagonals of a pentagon do not cross. The power of q appearing in the corresponding quasi-commutation relation is also combinatorially determined.

We now give an initial quantum seed for a quantum cluster algebra structure on $\mathbb{C}_q[\mathrm{Gr}(2,5)]$. For the initial quantum cluster we choose

$$\underline{\tilde{y}} = (\Delta_q^{15}, \boldsymbol{\Delta_q^{14}}, \boldsymbol{\Delta_q^{13}}, \Delta_q^{12}, \Delta_q^{23}, \Delta_q^{34}, \Delta_q^{45}).$$

This is a set of quasi-commuting variables by the above criterion: the corresponding diagonals of the pentagon are seen to be the five edges (in bijection with the coefficients) and two non-crossing diagonals, $(1,3)$ and $(1,4)$. That is, this cluster corresponds to a triangulation of the pentagon, as in the classical case (see e.g. [7]).

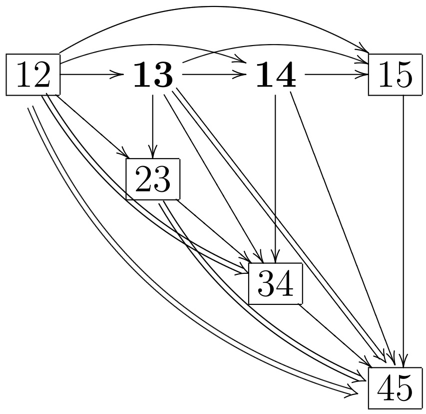
The corresponding quantum exchange matrix $\tilde{B}$ is equal to that for the well-known cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(2,5)]$ ([8]) and, along with its quiver $\Gamma(\tilde{B})$, is

$$\tilde{B} = \begin{pmatrix} -1 & 0 \\ \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma(\tilde{B}) = \begin{tikzcd} \boxed{12} \arrow[r] & \mathbf{13} \arrow[r] \arrow[d] & \mathbf{14} \arrow[r] \arrow[dd] & \boxed{15} \\ & \boxed{23} & & \\ & & \boxed{34} \arrow[uul] & \\ & & & \boxed{45} \arrow[uuul] \end{tikzcd}$$

where the quiver vertex corresponding to Δ_q^{ij} is labelled by ij. We see that this quantum cluster algebra is of type A_2, since the subquiver on the vertices **13** and **14** is an orientation of the Dynkin diagram of this type.

The quasi-commutation matrix $\tilde{L}$ and its quiver $\Gamma(\tilde{L})$ are

$$\tilde{L} = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 & 0 & 1 \\ 1 & \mathbf{0} & \mathbf{-1} & -1 & 0 & 1 & 1 \\ 1 & \mathbf{1} & \mathbf{0} & -1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 & -1 & 0 & 1 \\ -1 & -1 & -2 & -2 & -2 & -1 & 0 \end{pmatrix} \qquad \Gamma(\tilde{L}) =$$



These are compatible: $\tilde{B}^T\tilde{L} = 2I_2 \oplus 0_{2,5}$, where I_n is the $n \times n$ identity matrix and $0_{m,n}$ is the $m \times n$ zero matrix.

From this data and using the quantum exchange rule, we can write down the exchange relations and identify the remaining cluster variables. We know from the general theory of type A_2 cluster algebras that only three more cluster variables need to be identified. These are obtained from the mutations μ_1, μ_2 and $\mu_1 \circ \mu_2$ and the three exchange relations determining these are

$$\begin{aligned} \mu_1: &\qquad X_1X_1' = q^{-1}X_2X_4 + qX_3X_5 &\qquad \Rightarrow &\qquad X_1' = \boldsymbol{\Delta_q^{35}} \\ \mu_2: &\qquad X_2X_2' = qX_5X_7 + qX_1X_6 &\qquad \Rightarrow &\qquad X_2' = \boldsymbol{\Delta_q^{24}} \\ \mu_1 \circ \mu_2: &\qquad X_1X_1'' = qX_4X_7 + qX_2'X_3 &\qquad \Rightarrow &\qquad X_1'' = \boldsymbol{\Delta_q^{25}}. \end{aligned}$$

Here, X_i is the ith entry of the initial seed $\underline{\tilde{y}}$ and X_1', X_2' and X_1'' are the mutated variables. We see that the exchange relations are quantum Plücker relations. For example, the second of the equations above is

$$\boldsymbol{\Delta_q^{13}}\boldsymbol{\Delta_q^{24}} = q\Delta_q^{34}\Delta_q^{12} + q\boldsymbol{\Delta_q^{14}}\Delta_q^{23}.$$

We have also verified the remaining seven exchange relations, with the assistance of the computer program Magma.

Thus the complete set of cluster variables is

$$\{\boldsymbol{\Delta_q^{35}}, \boldsymbol{\Delta_q^{25}}, \boldsymbol{\Delta_q^{24}}, \boldsymbol{\Delta_q^{14}}, \boldsymbol{\Delta_q^{13}}\} \cup \{\Delta_q^{15}, \Delta_q^{12}, \Delta_q^{23}, \Delta_q^{34}, \Delta_q^{45}\}$$

where we have ordered the mutable cluster variables so that they are in bijection with the almost positive roots of the root system of type A_2 in

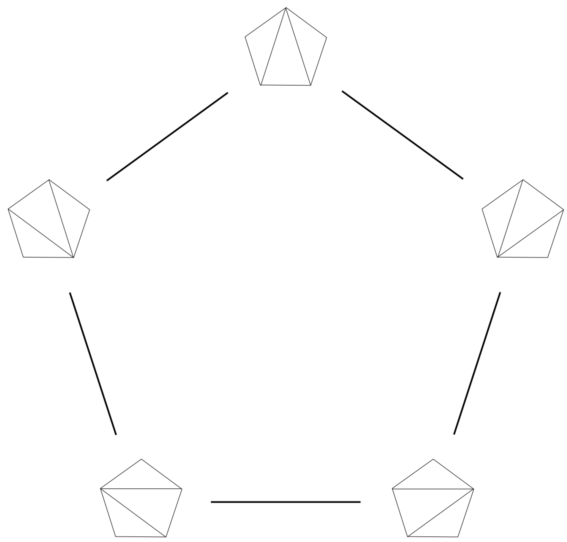


Figure 1: Exchange graph for cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(2,5)]$ and its quantum analogue.

the order $(\alpha_1, \alpha_1+\alpha_2, \alpha_2, -\alpha_1, -\alpha_2)$. We see that this set is equal to $\mathcal{P}_q$, the set of quantum Plücker coordinates which generates $\mathbb{C}_q[\mathrm{Gr}(2,5)]$. Hence $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ is a quantum cluster algebra.

We reproduce in Figure 1 the diagram in [9] showing the exchange graph in this case, with clusters identified with triangulations of the pentagon in the manner described previously. The top vertex corresponds to the initial cluster described here.

The unipotent radical

The unipotent radical $N^-_{\{2\}}$ associated to the above data is an affine space of dimension 6. Its quantized coordinate ring is identified with $\mathbb{C}_q[M(2,3)]$ which embeds into $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ via the map

$$A \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus A.$$

The quantum cluster algebra structure on $\mathbb{C}_q[N^-_{\{2\}}]$ may be obtained from that on $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ by noting that as a consequence of this map, $\mathbb{C}_q[N^-_{\{2\}}]$ is generated by the set of quantum minors $\mathcal{P}_q \setminus \{\Delta_q^{12}\}$. This reflects the construction of $\mathbb{C}_q[N^-_{\{2\}}]$ as a localisation of $\mathbb{C}_q[\mathrm{Gr}(2,5)]$ at (Δ_q^{12}).

Thus an initial quantum seed is given by

$$\underline{y} = (\Delta_q^{15}, \boldsymbol{\Delta_q^{14}}, \boldsymbol{\Delta_q^{13}}, \Delta_q^{23}, \Delta_q^{34}, \Delta_q^{45})$$

$$B = \begin{pmatrix} -1 & 0 \\ \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \\ 0 & -1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma(B) = \begin{tikzcd} \mathbf{13} \arrow[r] \arrow[d] & \mathbf{14} \arrow[r] \arrow[dd] & \boxed{15} \\ \boxed{23} & & \\ & \boxed{34} \arrow[uul] & \\ & & \boxed{45} \arrow[uuul] \end{tikzcd}$$

$$L = \begin{pmatrix} 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & \mathbf{0} & \mathbf{-1} & 0 & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{0} & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix} \qquad \Gamma(L) = \begin{tikzcd} \mathbf{13} \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dddrr] & \mathbf{14} \arrow[r] & \boxed{15} \\ \boxed{23} \arrow[ddrr, bend right] & & \\ & \boxed{34} \arrow[uur] \arrow[dr] & \\ & & \boxed{45} \end{tikzcd}$$

Notice that although B is given simply by deleting the row of $\tilde{B}$ labelled by 12, the matrix L is not related to $\tilde{L}$ in this way since the quasi-commutation relations are different in the localisation.

The quantum cluster algebra structure on $U_q(\mathfrak{n}^-_{\{2\}})$, a subalgebra of $U_q(\mathfrak{sl}_5)$, corresponds to that on $\mathbb{C}_q[N^-_{\{2\}}]$ via the bijection $\Delta_q^{ij} \mapsto q(D^{ij}_{q^{-1}})$, where $D^{ij}_{q^{-1}}$ is a q^{-1}-minor of the matrix

$$\begin{pmatrix} 0 & 1 & g_{12} & g_{13} & g_{14} \\ -1 & 0 & g_{22} & g_{23} & g_{24} \end{pmatrix}$$

and

$$\begin{aligned} g_{12} &= F_2 K_2 & g_{22} &= [\, F_1, F_2 \,]_q K_1 K_2 \\ g_{13} &= [\, F_3, F_2 \,]_q K_2 K_3 & g_{23} &= [\, F_3, [\, F_1, F_2 \,]_q \,]_q K_1 K_2 K_3 \\ g_{14} &= [\, F_4, [\, F_3, F_2 \,]_q \,]_q K_2 K_3 K_4 & g_{24} &= [\, F_4, [\, F_3, [\, F_1, F_2 \,]_q \,]_q \,]_q K_1 K_2 K_3 K_4 \end{aligned}$$

(F_i, K_i being the usual $U_q(\mathfrak{sl}_5)$-generators).

The mutable quantum cluster variables are
$\{q(D^{35}_{q^{-1}}), q(D^{25}_{q^{-1}}), q(D^{24}_{q^{-1}}), q(D^{14}_{q^{-1}}), q(D^{13}_{q^{-1}})\} = \{q(D^{35}_{q^{-1}}), qg_{24}, qg_{23}, g_{13}, g_{12}\}$
and the coefficients are

$$\{q(D^{15}_{q^{-1}}), q(D^{23}_{q^{-1}}), q(D^{34}_{q^{-1}}), q(D^{45}_{q^{-1}})\} = \{g_{14}, qg_{22}, q(D^{34}_{q^{-1}}), q(D^{45}_{q^{-1}})\}$$

Again the mutable cluster variables are in bijection with the almost-positive roots of A_2, in the same order as before. One can show that the g_{ij} generate $U_q(\mathfrak{n}^-_{\{2\}})$ and hence this algebra is also a quantum cluster algebra.

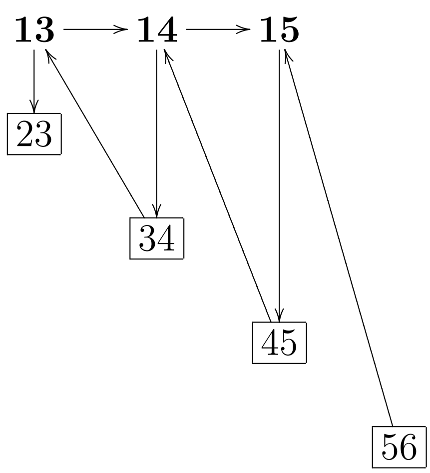
Example: the Grassmannians $\mathrm{Gr}(2, n)$ and their Schubert cells

With Stéphane Launois we have extended the above to the quantized coordinate rings $\mathbb{C}_q[\mathrm{Gr}(2, n)]$ for $n \geq 3$, showing that they are quantum cluster algebras of type A_{n-3}. From this, we have also obtained quantum cluster algebra structures on the quantum Schubert cells of these Grassmannians. The quantum Schubert cell associated to the partition (t, s) ($t \geq s$, $t + s \leq 2n - 2$) is of quantum cluster algebra type A_{s-1}, independent of t.

It is expected that the quantum Grassmannians $\mathbb{C}_q[\mathrm{Gr}(k, n)]$ are quantum cluster algebras for any k and n but even classically the only finite-type cases are $n = 2$ and $(3, 6)$, $(3, 7)$ and $(3, 8)$ (considering only $k \leq n/2$) and so it is likely that some geometric arguments will be required.

Example: a corank 2 example in type A_4

We have also explicitly calculated a quantum cluster algebra structure on $U_q(\mathfrak{n}^-_{\{1,2\}}) \subseteq U_q(\mathfrak{sl}_5)$. This example is of cluster algebra type A_3; the quiver describing the exchange matrix in both the classical and quantum cases is



Questions and a conjecture

The examples we have given are clearly closely related. We ask a question generalizing Gautam's Question (B) in [10]:

Question 1. Is there a method for obtaining (quantum) cluster algebra structures on (quantizations of) $\mathbb{C}[N_D^-]$ and $\mathbb{C}[G/P_J]$ from (quantum) cluster algebra structures on $\mathbb{C}[N_I^-]$ and $\mathbb{C}[G/B]$?

Gautam proposed that one should carry out "elementary operations" such as mutation and freezing of vertices and demonstrated a positive answer in type G_2. We concur that a general construction of this type would be desirable and remark that we have seen traces of such a phenomenon in the case of quantum Schubert cells.

The exchange matrices B and compatible quasi-commutation matrices L appearing in all our examples have the property that the diagonal block appearing in $B^T L$ is precisely $2I_m$ for some m.

Question 2. What is the significance of this? Is it encoding some Lie-theoretic phenomenon?

More precisely, in the double Bruhat cell case the quasi-commutation matrix encodes a Poisson structure. Poisson structures on Lie groups are in precise correspondence with Lie coalgebra structures on the associated Lie algebra: Poisson-Lie groups correspond to Lie bialgebras. We note that the classical analogues $\mathfrak{n}_D^-$ of the braided enveloping algebras $U_q(\mathfrak{n}_D^-)$ are braided-Lie bialgebras ([11]) and expect that the braided-Lie cobrackets on $\mathfrak{n}_D^-$ should be reflected in the structure of $U_q(\mathfrak{n}_D^-)$.

Question 3. Do the quasi-commutation matrices in our examples relate to Poisson structures?

Finally, there is a natural conjecture to make:

Conjecture. *For any sub-root datum $\mathcal{C}' \subseteq \mathcal{C}$ of a finite-type root datum $\mathcal{C}$, the quantized coordinate ring of the unipotent radical $\mathbb{C}_q[N_D^-]$ admits a quantum cluster algebra structure, quantizing that of Geiß, Leclerc and Schröer on $\mathbb{C}[N_D^-]$. Furthermore, this quantum cluster algebra structure lifts to one on the quantum partial flag variety $\mathbb{C}_q[G/P_J]$ and induces one on the dual $U_q(\mathfrak{n}_D^-)$.*

Note that even classically the equality of the algebra generated by the cluster variables with the coordinate algebra is still conjectural for some cases, although it is expected to hold in general. The above conjecture should certainly be true when the cluster algebra structure on $\mathbb{C}[N_D^-]$ is of finite type.

We would also like an associated categorification, perhaps coming from either $\Lambda(\mathcal{C})$-mod (Λ the preprojective algebra) with some extra structure or from deformed preprojective algebras. One could then hope for new information about canonical bases, just as Geiß, Leclerc and Schröer have related their cluster algebra structures to the (dual) semicanonical basis.

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