Sheaves on subanalytic sites and $\mathcal{D}$-modules

Luca Prelli

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Categories

Definition: A category $\mathcal{C}$ is the data of a set $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$, and for any $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, with a composition $\circ$ which is associative and satisfying $f \circ \mathrm{id} = f$ and $\mathrm{id} \circ g = g$.

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Examples of categories

1. Set: the objects are sets and morphisms are maps between sets.
2. $\mathrm{Mod}(k)$ (k a field): the objects are k-vector spaces and morphisms are linear maps.
3. $\mathrm{Op}(X)$ (X a topological space): the objects are the open subsets of X and the morphisms are the inclusions.

Functors

Definition: Given two categories $\mathcal{C}, \mathcal{C}'$, a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is the data of a morphism

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and for each $X, Y \in Ob(\mathcal{C})$, a morphism

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Definition: Two categories are equivalent if there is a functor F such that F_o is a bijection between the isomorphism classes of objects and F_m is a bijection between the set of morphisms.

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Satisfying the following gluing conditions. Let U be open and let $\{U_j\}_{j\in J}$ be a covering of U. We have the exact sequence

$$0 \to F(U) \to \prod_{j\in J} F(U_j) \to \prod_{j,k\in J} F(U_j \cap U_k)$$

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- if $s_j \in \Gamma(U_j; F)$ such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \Gamma(U; F)$ (i.e. $s|_{U_j} = s_j$)

Examples

Let us consider

$$\begin{aligned} \mathbb{R}_X : \text{Open sets of } X &\to \mathrm{Mod}(\mathbb{R}) \\ U &\mapsto \Gamma(U; \mathbb{R}_X) = \{\text{constant functions on } U\} \end{aligned}$$

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- If s is zero on a covering of U then $s = 0$.
- For example, let $X = \mathbb{R}$, $U_1 = (1,2)$, $U_2 = (2,3)$. We have $U_1 \cap U_2 = \varnothing$. The constant functions $s_1 = 0$ on U_1 and $s_2 = 1$ on U_2 do not glue on a constant function on $U_1 \cup U_2$.

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$\Rightarrow$ The correspondence
$U \mapsto \Gamma(U;\mathbb{R}_X) = \{\text{constant functions on } X\}$ does not define a sheaf on X. We have to consider locally constant functions.

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- If s is a continuous function and s is zero on a covering of U then $s = 0$.
- If $\{s_i\}$ are continuous functions on a covering $\{U_i\}$ of U, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists s continuous on U with $s = s_i$ on each U_i.

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- For example, let $X = \mathbb{R}$, $U_n = (-n, n)$, $n \in \mathbb{N}$, and $s_n \colon U_n \to \mathbb{R}$, $x \mapsto x^2$. Then s_n is bounded on U_n for each $n \in \mathbb{N}$, but $x \mapsto x^2$ is not bounded on $\mathbb{R}$.

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More Examples

Sheaves: holomorphic functions, $\mathcal{C}^\infty$ functions , distributions.

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Sheaves: holomorphic functions, $\mathcal{C}^\infty$ functions , distributions.
Not sheaves: L^2 functions, tempered distributions. In fact they do not satisfy gluing conditions.

If we consider “less open subsets” and “less coverings” they may become sheaves. We need the notion of site.

Fibers

Let $F \in \mathrm{Mod}(k_X)$ we define the fiber of F at x as

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Moreover, given $U_1, U_2 \ni x$ and $f_i \in U_i$, we have $f_1 \equiv f_2$ in F_x if $f_1 = f_2$ on a neighborhood of x $W \subset U_1 \cap U_2$.

Fibers

Two sheaves F, G are isomorphic if

$$F_x \simeq G_x$$

for any $x \in X$. More generally a sequence of sheaves

$$0 \to F' \to F \to F'' \to 0$$

is exact if the sequence

$$0 \to F'_x \to F_x \to F''_x \to 0$$

is exact in $\mathrm{Mod}(k_X)$.

Topological sites

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Then $F : \mathcal{T} \to \mathrm{Mod}(k)$ is a sheaf on $X_{\mathcal{T}}$ if for each $U \in \mathcal{T}$ and each $\{U_j\}_{j \in J} \in \mathrm{Cov}(U)$ we have the exact sequence

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- If $\{s_i\}$ are bounded on a finite covering $\{U_i\}$ of U, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists s bounded on U with $s = s_i$ on each U_i.

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defines a sheaf on $X_{\mathcal{T}}$.

The general case

Let X be a topological space and consider a family of open subsets $\mathcal{T}$ satisfying:

$$\begin{cases} \text{(i) } U, V \in \mathcal{T} \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite numbers of connected components } \forall U, V \in \mathcal{T}, \\ \text{(iii) } \mathcal{T} \text{ is a basis for the topology of } X. \end{cases}$$

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- $\mathrm{Cov}(U)$ (coverings of $U \in \mathrm{Op}(X_{\mathcal{T}})$): finite coverings of U

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2. $\mathcal{T}$={open relatively compact subanalytic subsets of a real analytic manifold}, the subanalytic site X_{sa}.
3. $\mathcal{T}$={open definable subsets of N^n}, given an O-minimal structure $(N, <, \dots)$, the site DTOP.

Construction of sheaves on $X_{\mathcal{T}}$

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Then F is a sheaf on $X_{\mathcal{T}}$.

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of tempered distribution over a real analytic manifold X. This is not a sheaf with the usual topology.
For example, if $X = \mathbb{R}$, we can find tempered distributions s_n on $\{\frac{1}{n} < x < 1\}$, $n \in \mathbb{N}$ which do not glue to a tempered distribution s on $\{0 < x < 1\}$.

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S. Łojasiewicz *Sur le problème de la division*, Studia Mathematica **8** pp. 87-136 (1959).

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This implies that $U \mapsto \mathcal{D}b^t_X(U)$ is a sheaf on the subanalytic site X_{sa}.

Tempered holomorphic functions

Let X be a complex manifold and let $U \subset X$ be a relatively compact subanalytic open subset, f holomorphic on U is tempered if $\exists\, M, C > 0$ such that

$$|f(z)| \leq \frac{C}{\mathrm{dist}(z, \partial U)^M}\ .$$

Fibers

In the case of subanalytic sheaves we do not have the notion of fibers in the usual sense, i.e. if we consider

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i.e. there are $F \not\simeq G$ even if $F_x \simeq G_x$ $\forall x \in X$.

Example: Let $X = \mathbb{R}$ and consider the sheaves $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}^b_{\mathbb{R}}$. Then $\mathcal{C}_{\mathbb{R},x} \simeq \mathcal{C}^b_{\mathbb{R},x}$ $\forall x \in \mathbb{R}$. Indeed, any continuous function f in $(x-\varepsilon, x+\varepsilon)$, $\varepsilon > 0$ is bounded in $(x-\varepsilon/2, x+\varepsilon/2)$.

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We need to consider more points.

Spectral topology

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A neighborhood of an ultrafilter α is a globally subanalitic open subset U contained in α.
We call $\widetilde{X}$ the associated topological space. In $\widetilde{X}$ any covering of a relatively compact subanalytic open subset has a finite subcover.

Example

For example, the points of $\widetilde{\mathbb{R}}$ are the following. Let $x \in \mathbb{R}$

1. $\{S \,\text{subanalytic}, \, S \supseteq x\}$ (the point x)
2. $\{S \,\text{subanalytic}, \, S \supseteq (x, x+\varepsilon), \ \varepsilon > 0\}$ (the point x^+)
3. $\{S \,\text{subanalytic}, \, S \supseteq (x-\varepsilon, x), \ \varepsilon > 0\}$ (the point x^-)

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Thanks to these new points we can distinguish $\mathcal{C}_{\mathbb{R}}$ from $\mathcal{C}^b_{\mathbb{R}}$ on $\widetilde{\mathbb{R}}$. For example let $f = x^{-1}$. Then $f \notin \mathcal{C}^b_{\mathbb{R}}(0, \varepsilon)$ $\forall \varepsilon > 0$.

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Topological and subanalytic sheaves

Theorem:

Let X be a real analytic manifold. The categories $\mathrm{Mod}(k_{X_{sa}})$ and $\mathrm{Mod}(k_{\widetilde{X}})$ are equivalent.

Hence, if we want to work on fibers on X_{sa}, we have to consider the topological space $\widetilde{X}$.

Operations

Theorem:

Let $f : X \to Y$ be a morphism of real analytic manifolds. The six Grothendieck operations $\mathcal{H}om$, $\otimes$, f_*, f^{-1}, $f_{!!}$, $f^!$ are well defined for subanalytic sheaves.

L. PRELLI *Sheaves on subanalytic sites*, Rendiconti del Seminario Matematico dell'Università di Padova Vol. **120** (2008).

The ring of differential operators

Let X be a complex analytic manifold. We denote by $\mathcal{D}_X$ the sheaf of rings of differential operators. Locally, a section of $\Gamma(U; \mathcal{D}_X)$ may be written as $P = \sum_{|\alpha| \leq m} a_\alpha(z)\partial_z^\alpha$ with $a_\alpha(z)$ holomorphic on U.
We denote by $\mathrm{Mod}(\mathcal{D}_X)$ the sheaf of $\mathcal{D}_X$-modules.

Complex of solutions

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Definition: If U is open, $\mathcal{F}$ a $\mathcal{D}_X$-module, P a differential operator, $\mathcal{S}ol_{\mathcal{F}}(P)$ on U is the complex

$$\Gamma(U;\mathcal{F}) \xrightarrow{P} \Gamma(U;\mathcal{F})$$

$$\begin{aligned} H^0(U;\mathcal{S}ol_{\mathcal{F}}(P)) &= \{s \in \Gamma(U;\mathcal{F}),\ Ps = 0\} = \ker P \\ H^1(U;\mathcal{S}ol_{\mathcal{F}}(P)) &= \Gamma(U;\mathcal{F})/P\Gamma(U;\mathcal{F}) = \mathrm{coker} P \end{aligned}$$

Complex of solutions

The sheaf $\mathcal{O}_X$ of holomorphic functions has a structure of $\mathcal{D}_X$-module.

Definition: If U is open, $\mathcal{F}$ a $\mathcal{D}_X$-module, P a differential operator, $\mathcal{S}ol_{\mathcal{F}}(P)$ on U is the complex

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Definition: P_1 and P_2 are equivalent if for any $\mathcal{F}$ $\ker P_1 \simeq \ker P_2$ and $\mathrm{coker} P_1 \simeq \mathrm{coker} P_2$ (i.e. $\mathcal{S}ol_{\mathcal{F}}(P_1)$ and $\mathcal{S}ol_{\mathcal{F}}(P_2)$ are quasi-isomorphic).

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Let $\alpha \in \mathbb{C}$, and consider the operators

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the above morphisms induce an isomorphism between the homogeneous solutions $\ker P_\alpha$ and $\ker P_{\alpha+1}$.

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Moreover one can prove that the above morphisms induce an isomorphism between $\mathrm{coker} P_\alpha$ and $\mathrm{coker} P_{\alpha+1}$ (i.e. the complexes $\mathcal{S}ol_{\mathcal{F}}(P_\alpha)$ and $\mathcal{S}ol_{\mathcal{F}}(P_{\alpha+1})$ are quasi-isomorphic).

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Hence P_α and $P_{\alpha+1}$ are equivalent.

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Let $f \in \mathcal{O}_X(U)$.

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Then $z\partial_Z$ and ∂_Z are not equivalent.

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Hence $z(z\partial_z + 1)$ and $z^2\partial z + 1$ are not equivalent (even if the holomorphic solutions are).

Equivalence for regular operators

Definition: $P = \sum_{\alpha \leq n} a_\alpha(z)\partial^\alpha$, $a_\alpha(0) \neq 0$, is regular at 0 if for each $j \leq n$, $n - \mathrm{ord}_0(a_n) \geq j - \mathrm{ord}_0(a_j)$.

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Theorem: Let P and Q be regular at 0. The following are equivalent.

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In particular the holomorphic solutions are sufficient to establish if two regular equations are equivalent.

Subanalytic sheaves and solutions

The sheaf $\mathcal{O}^t_X$ of tempered holomorphic functions has a structure of $\rho_!\mathcal{D}_X$-module. ($\Gamma(U; \rho_!\mathcal{D}_X)$ are differential operators $\sum_{|\alpha|\leq m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in $\overline{U}$)

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Let us consider the operators $z^2\partial_z + 1$ and $z^3\partial_z + 2$. Their solutions are respectively $\exp(z^{-1})$ and $\exp(z^{-2})$.

Theorem (G. Morando): There exists an open subanalytic U such that $\exp(z^{-1}) \in \mathcal{O}^t_X(U)$ and $\exp(z^{-2}) \notin \mathcal{O}^t_X(U)$.

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In particular for such U we have

$$\begin{aligned} H^0(U; \mathcal{S}ol_{\mathcal{O}^t_X}(z^2\partial_z + 1)) &\simeq \mathbb{C} \cdot \exp(z^{-1}) \\ H^0(U; \mathcal{S}ol_{\mathcal{O}^t_X}(z^3\partial_z + 2)) &\simeq 0. \end{aligned}$$

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Hence thanks to tempered holomorphic solutions we can distinguish irregular differential operators which cannot be distinguished with holomorphic solutions.

Sheaves on subanalytic sites and $\mathcal{D}$-modules

Luca Prelli

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