O-minimal sheaves and applications

(Joint work with L. Prelli)

Mário Edmundo
UAb & CMAF/UL, PT

Toronto, May 4 - 8, 2009

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The Setting

Given an o-minimal structure

$$\mathcal{M} = (M, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}, <)$$

we have:

- the category Def of definable spaces with continuous definable maps.
- the geometry of Def is called o-minimal geometry.

Examples (Special Cases of O-minimal Geometry)

- $\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, <)$ - semi-algebraic geometry (includes real algebraic geometry);
- $\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, (f)_{f\in \mathrm{an}}, <)$ - globally sub-analytic geometry;

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General motivation of our work

Develop sheaf theory in the category Def:

Inspired by:

- Verdier (locally compact topological spaces);
- Grothendieck (étale framework);
- Delfs (real algebraic geometry).
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What is an o-minimal sheaf?

Let X be an object of Def and k a field. An o-minimal sheaf of k-vector spaces on X a contravariant functor:

$$\begin{aligned} F : \mathrm{Op}(X_{\mathrm{def}}) &\to Mod(k) \\ U &\mapsto \Gamma(U; F) \\ (V \subset U) &\mapsto (F(U) \to F(V)) \\ & \qquad\qquad s \mapsto s_{|V} \end{aligned}$$

where X_{def} is the o-minimal site on X. Satisfying the following gluing conditions: for $U \in \mathrm{Op}(X_{\mathrm{def}})$ and $\{U_j\}_{j \in J} \in \mathrm{Cov}(U)$ we have the exact sequence

$$0 \to F(U) \to \Pi_{j \in J} F(U_j) \to \Pi_{j,k \in J} F(U_j \cap U_k)$$

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What is the o-minimal site on X?

Let X be an object of Def. The o-minimal site X_{def} on X is the data consisting of:

- The category
$$\mathrm{Op}(X_{\mathrm{def}})$$
of open definable subsets of X with inclusions;
- The collection of admissible coverings
$$\mathrm{Cov}(U), \;\; U \in \mathrm{Op}(X_{\mathrm{def}})$$
such that $\{U_j\}_{j \in J} \in \mathrm{Cov}(U)$ if $\{U_j\}_{j \in J}$ covers U and has a finite sub-cover.

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Replacing the o-minimal site by the o-minimal spectrum

It is convenient to replace the o-minimal site X_{def} by the o-minimal spectrum $\widetilde{X}$ of X:

- $\widetilde{X}$ is the set of ultrafilters of definable subsets of X equipped with the topology generated by the open subsets of the form $\widetilde{U}$ where $U \in \mathrm{Op}(X_{\mathrm{def}})$.

This tilde operation determines a functor

$$\mathrm{Def} \to \widetilde{\mathrm{Def}}.$$

Example

If R is a r.c.f and X an affine real algebraic variety over R with coordinate ring $R[X]$, then $\widetilde{X} \simeq \mathrm{Specr} R[X]$.

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Properties of the o-minimal spectrum

Proposition (Pillay)

The o-minimal spectrum $\widetilde{X}$ of a definable space X is T_0, quasi-compact and a spectral topological spaces, i.e:

- it has a basis of open quasi-compact subsets closed under finite intersections.
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Canonical isomorphism

The tilde functor $\mathrm{Def} \longrightarrow \widetilde{\mathrm{Def}}$ determines morphisms of sites

$$\nu_X : \widetilde{X} \longrightarrow X_{\mathrm{def}}$$

given by the functor $\nu_X^t : \mathrm{Op}(X_{\mathrm{def}}) \longrightarrow \mathrm{Op}(\widetilde{X}) : U \mapsto \widetilde{U}$.

Theorem (E, Peatfield and Jones)

The functor $\mathrm{Def} \longrightarrow \widetilde{\mathrm{Def}}$ induces an isomorphism of categories

$$\mathrm{Mod}(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{Mod}(k_{\widetilde{X}}) : F \mapsto \widetilde{F},$$

where $\mathrm{Mod}(k_{\widetilde{X}})$ is the category of sheaves of k-modules on the topological space $\widetilde{X}$.

It is the inverse image ν_X^{-1} and its inverse is the direct image ν_{X*}.

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Canonical isomorphism in derived categories

The canonical isomorphism extends to the derived categories

$$\mathrm{D}^*(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^*(k_{\widetilde{X}}) : I \mapsto \widetilde{I}$$

where $\mathrm{D}^*(k_{\widetilde{X}}) = \mathrm{D}^*(\mathrm{Mod}(k_{\widetilde{X}}))$ and $(* = b, +, -)$.

Corollary

The functors

$$\mathrm{RHom}_{k_{X_{\mathrm{def}}}}(\bullet, \bullet) : \mathrm{D}^-(k_{X_{\mathrm{def}}})^{\mathrm{op}} \times \mathrm{D}^+(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^+(k),$$

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commute with the tilde functor.

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commute with the tilde functor.

Canonical isomorphism and proofs

So we can develop o-minimal sheaf cohomology by setting

$$H^*(X; F) := H^*(\widetilde{X}; \widetilde{F})$$

where X is a definable space and F is a sheaf in $\mathrm{Mod}(k_{X_{\mathrm{def}}})$.
Moreover, we can proof properties of our operations on o-minimal sheaves by going to the tilde world and then came back:

Theorems (E, Peatfield and Jones)

- Vanishing Theorem.
- Vietoris-Begle Theorem.
- Eilenberg-Steenrod Axioms.

Comments about assumptions and proof technique in the tilde world...

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More proofs

Theorems (E, Peatfield and Jones + E, L. Prelli)

- Base Change Theorem:

$$g^{-1}Rf_*F \simeq Rf'_*(g'^{-1}F).$$

- Projection Formula:

$$Rf_*F \otimes_{k_{X_{\mathrm{def}}}} G \simeq Rf_*(F \otimes_{k_{X_{\mathrm{def}}}} f^{-1}G).$$

- Universal Coefficients Formula:

$$R\Gamma(X; \underline{m}) \simeq R\Gamma(X; \underline{k}) \otimes_k m.$$

- Künneth Formula:

$$R\Gamma(X \times Y; \underline{l} \otimes_{k_{X_{\mathrm{def}}}} \underline{m}) \simeq R\Gamma(X; \underline{l}) \otimes_k R\Gamma(Y; \underline{m}).$$

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Families of definably normal supports

Suppose that X is an object of Def which is definably normal and definably locally compact. Then the collection c of definably compact subsets of X is a family of definably normal supports, i.e:

- every closed definable subset of a member of c is in c;
- c is closed under finite unions;
- each element of c is definably normal;
- each element of c has a closed definable neighborhood which is in c.

(These assumptions will be assumed below.)

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c-soft sheaves

We say that $F \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$ is c-soft if the restriction

$$\Gamma(X; F) \to \Gamma(S; F_{|S})$$

is surjective for every $S \in c$.

Theorem (E, L. Prelli)

The full additive subcategory of $\mathrm{Mod}(k_{X_{\mathrm{def}}})$ of c-soft k-sheaves is:

- $\Gamma_c(X; \bullet)$-injective;
- stable under filtrant inductive limits;
- stable under $\bullet \otimes_{k_{X_{\mathrm{def}}}} F$ for every $F \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$.

Note: X has cohomological c-dimension bounded by $\dim X$, i.e., $H^q_c(X; F) = 0$ for all $q > \dim X$.

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- $\Gamma_c(X; \bullet)$-injective;
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Note: X has cohomological c-dimension bounded by $\dim X$, i.e., $H^q_c(X; F) = 0$ for all $q > \dim X$.

c-soft sheaves

We say that $F \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$ is c-soft if the restriction

$$\Gamma(X; F) \to \Gamma(S; F_{|S})$$

is surjective for every $S \in c$.

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Sheaves of linear forms

For $F \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$ we define a presheaf $F^\vee$ by

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- $G^\vee \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$;
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Verdier Duality

Passing to the derived category we obtain:

Theorem (E, L. Prelli)

There exists $\mathcal{D}^*$ in $\mathrm{D}^+(k_{X_{\mathrm{def}}})$ and a natural isomorphism

$$\mathrm{RHom}_{k_{X_{\mathrm{def}}}}(\mathcal{F}^*, \mathcal{D}^*) \simeq \mathrm{RHom}_k(R\Gamma_c(X, \mathcal{F}^*), k)$$

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The cohomological k-sheaves $\mathcal{H}^{-p}\mathcal{D}^*$ are the sheafifications of the presheaves

$$U \mapsto H_c^p(U; k_X)^\vee.$$

For $p =$ cohomological c-dimension of X these are k-sheaves. Hence:

Theorems (E.-L.-Prelli)

Let X be definable manifold of dimension n.

- If X has an orientation k-sheaf $\mathcal{O}r_X$, then

$$H^p(X; \mathcal{O}r_X) \simeq H_c^{n-p}(X; k)^\vee.$$

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Let X be a definable manifold of dimension n. We say that X has an orientation k-sheaf if for every $U \in \mathrm{Op}(X_{\mathrm{def}})$ there exists $\{U_j\}_{j\in J} \in \mathrm{Cov}(U)$ such that for each j we have

$$H^p_c(U_j; k_X) = \begin{cases} k & \text{if} \quad p = n \\ 0 & \text{if} \quad p \neq n. \end{cases}$$

If X has an orientation k-sheaf, we call the k-sheaf $\mathcal{O}r_X$ on X with sections

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When do orientation k-sheaves exist?

Suppose that

$$\mathcal{M} = (M, 0, +, <, (c)_{c \in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}})$$

is an o-minimal expansion of an ordered group. Then every definable manifold has an orientation k-sheaf.

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- Do we have a duality compatible with o-minimal singular homology?
- Orientability with respect to cohomology is compatible with orientability with respect to homology?

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Suppose that $\mathcal{M}$ is an o-minimal expansion of a real closed field and X is a Hausdorff definable manifold of dimension n.

Theorem

If $L \subseteq K \subseteq X$ are closed definable sets with $K - L$ closed in $X - L$, then there is an isomorphism

$$H_c^q(K \setminus L; k) \longrightarrow H_{n-q}(X \setminus L, X \setminus K; k)$$

for all $q \in \mathbb{Z}$ which is natural with respect to inclusions.

Corollary

Hence, X is k-orientable with respect to homology if and only if X is k-orientable with respect to cohomology.

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What is the main application?

We assume that $\mathcal{M} = (M, <, \dots)$ is a sufficiently saturated o-minimal structure with definable Skolem functions.

Theorem (E and Terzo)

Let G be a $\mathbb{Z}/q\mathbb{Z}$-orientable, definably connected, definably compact, definable group, where q is some sufficiently large prime number. Then there exists a smallest type definable normal subgroup G^{00} of G of bounded index such that G/G^{00} with the logic topology is a connected, compact, Lie group. Moreover, the following hold:

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Without the orientability assumption, this is:

- Pillay's conjecture for definably compact groups;
- A non-standard version of Hilbert's 5º problem for locally compact groups.

Previously known cases:

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