Asymptotic Analysis for Optimal Investment with Two Risky Assets and Transaction Costs

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History

- Merton, J. Econ, 1971
  - Optimal consumption and investment problem in continuous time.
- Magill & Constantinides, JET, 1976
  - Proportional transaction costs in Merton's model.
- Davis & Norman, Mathematics of Operations Research, 1990
  - Rigorous treatment of Magill & Constatinides model
  - Existence and uniqueness of solution to the HJB equation
  - Shape of the optimal policy.
- Shreve & Soner, Annals of Applied Probability, 1994
  - Viscosity solution analysis of Magill & Constantinides model
  - Smoothness of the value function.
- Whalley & Wilmott, Mathematical Finance 1997
  - Pricing an option
  - Asymptotic expansion of the value function in powers of $\lambda^{\frac{1}{3}}$.

History (cont)

- Janeček & Shreve, Finance and Stochastics, 2004
  - Model with one stock
  - Viscosity solution approach to compute the loss in asymptotic expansion of the value function.
- Atkinson & Ingpochai J. of Comp Fin. 2007
  - Multiple assets
  - Loss in asymptotic expansion of the value function
  - Asymptotically correlated assets.

Model Definition

- Buyer of a Futures contract receives changes in futures price.
- Change in futures price of futures contract of type i

$$dF_i(t) = \mu_i dt + \sigma_i dB_i(t),$$

where $< B_1, B_2 >_t = \rho t$.
- Number of futures contracts of type i held at time t

$$Y_i(t) = y_i + L_i(t) - M_i(t).$$

- Change in cash held at time t

$$dX(t) = \sum_{i=1}^{2} Y_i(t) dF_i(t) - \sum_{i=1}^{2} \lambda(dL_i(t) + dM_i(t)) \\ +X(t)(r - c(t))dt.$$

- Initial endowment

$$X(0-) = x. \quad Y_i(0-) = y_i.$$

- $\mathcal{S}_v = \{(y_1, y_2, x) : x - \lambda|y_1| - \lambda|y_2| > 0\}$.
- The strategy (c, L_i, M_i) is admissible for (y_1, y_2, x), if $(Y_1(t), Y_2(t), X(t)) \in \overline{\mathcal{S}_v}$ for all $t \geq 0$.
- $U(c) \triangleq \frac{c^p}{p}, \ c \geq 0, \ 0 < p < 1$.
- Max the expected integral of the discounted utility of consumption

$$v(y_1, y_2, x) = \sup_{\substack{\text{Admissible} \\ \text{strategies}}} E\left[\int_0^\infty e^{-\beta t} U(X(t)c(t))dt\right].$$

Properties of the value function

- $v\Big|_{\partial\mathcal{S}_v} = 0.$
- v is homogeneous of degree p:
  $\forall\alpha > 0: \; v(\alpha y_1, \alpha y_2, \alpha x) = \alpha^p v(y_1, y_2, x).$
- Reduction of variables

$$u(z_1, z_2) \triangleq v(z_1, z_2, 1),$$
$$v(y_1, y_2, x) = x^p v\left(\frac{y_1}{x}, \frac{y_1}{x}, 1\right) = x^p u(z_1, z_2), \;\; z_i = \frac{y_i}{x}.$$

Merton's Model results

$\lambda = 0$

Optimal investment proportion: $\theta_i = \frac{\mu_i \sigma_j - \rho \mu_j \sigma_i}{(1-p)\sigma_i^2 \sigma_j (1-\rho^2)}$,

$$A = \frac{\beta - rp}{1-p} - \frac{p}{1-p} \sum_{i=1}^{2} \mu_i \theta_i + \frac{p}{2} \left(\sigma_1^2 \theta_1^2 + \sigma_1^2 \theta_1^2 + 2\rho \sigma_1 \sigma_2 \theta_1 \theta_2 \right).$$

Standing Assumption

$A > 0$.

$\lambda = 0$

$$u(z_1, z_2) = \frac{1}{p} A^{p-1},$$

Optimal consumption proportion $c(t) = A$.

HJB equation ($\lambda > 0$)

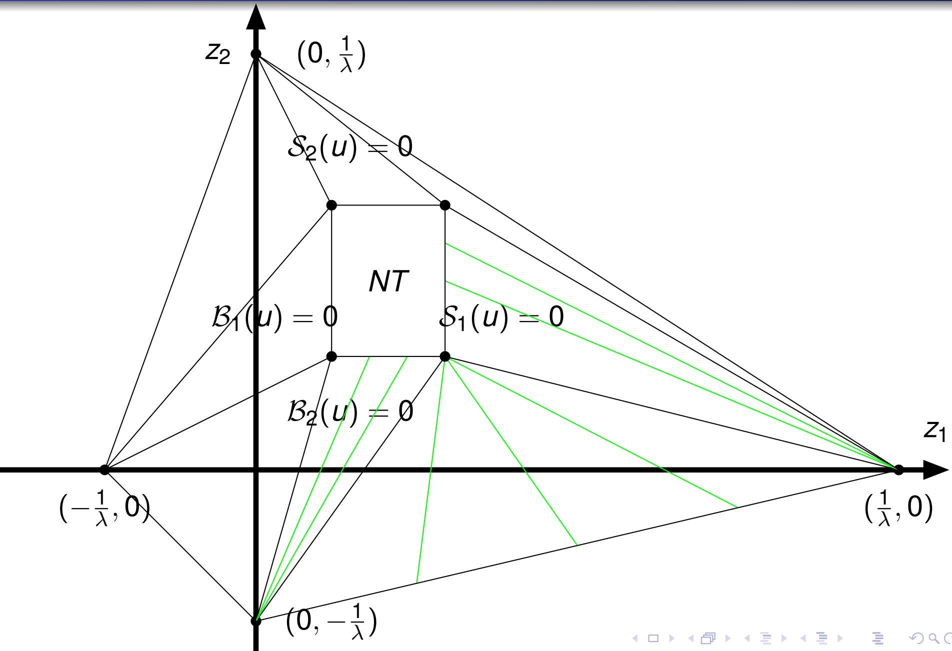
$$\widetilde{U}(\tilde{c}) \triangleq \sup_{c>0} \{U(c) - c\tilde{c}\} = \frac{1-p}{p}\tilde{c}^{-\frac{p}{1-p}}.$$

HJB equation

$$\min\left[\mathcal{D}(u) - \widetilde{U}\left(pu - z_1 u_1 - z_2 u_2\right), \mathcal{B}_1(u), \mathcal{S}_1(u), \mathcal{B}_2(u), \mathcal{S}_2(u)\right] = 0.$$

- What is $\mathcal{D}(u)$?
  - Second order diffusion operator.
  - Diffusion in radial direction only.
- What are $\mathcal{B}_1(u), \mathcal{S}_1(u), \mathcal{B}_2(u), \mathcal{S}_2(u)$?
  - First order differential operators.
  - Their characteristics correspond to pure trades.

Division of solvency region



Main Theorem 1, when $\rho = 0$

Theorem

Assume $0 < p < 1,\ A > 0,\ \lambda > 0$ *and* $\rho = 0$. *Fix* $(z_1, z_2) \in \mathcal{S}_u$, *then the value function is*

$$u(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2 \lambda^{\frac{2}{3}} + O(\lambda),$$

where

$$\gamma_2 = A^{p-2} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32}(1-p)\sigma_i^2 \theta_i^4 \left(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right)^2}.$$

Construction of viscosity sub- and super-solutions, when $\rho = 0$

Lemma

Assume $0 < p < 1,\ A > 0,\ \lambda > 0$ *and* $\rho = 0$. *Then there exist* $NT^{\pm}$ *regions, such that if inside we define:*

$$w^{\pm}(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2 \lambda^{\frac{2}{3}} \pm M\lambda$$
$$- \sum_{i=1}^{2} \frac{A^{p-1}}{\nu_i} \left(\frac{3}{2}(z_i - \theta_i)^2 \lambda^{\frac{2}{3}} - \frac{1}{\nu_i^2}(z_i - \theta_i)^4 + \frac{3}{2} B (z_i - \theta_i)^2 \lambda^{\frac{4}{3}} \right),$$

and define $w^{\pm}$ *in the rest of the solvency region* $\mathcal{S}_u$ *using characteristics. Then* $w^{\pm}$ *is a super/sub-solution of the HJB equation with zero boundary condition.*

Heuristics, when $\rho \neq 0$

What goes wrong?

If one assumes that the value function u has a power expansion and is $C^2(\mathcal{S}_u)$, then the system of equations obtained by equating the derivatives across the boundary of the NT region is inconsistent.

An auxiliary model, when $\rho > 0$

Consider a new set of futures contracts:

$$\tilde{F}_1(t) = F_1(t), \quad \tilde{F}_2(t) = F_2(t) - \rho\frac{\sigma_2}{\sigma_1}F_1(t).$$

To write them in the original form $\tilde{F}_i(t) = \tilde{\mu}_i t + \tilde{\sigma}_i \tilde{B}_i(t)$, define:

$$\tilde{B}_1(t) = B_1(t), \quad \tilde{B}_2(t) = \frac{1}{\sqrt{1-\rho^2}}\left(B_2(t) - \rho B_1(t)\right).$$

$$\begin{aligned}
\tilde{\sigma}_1 &= \sigma_1, \quad \tilde{\sigma}_2 = \sigma_2\sqrt{1-\rho^2}, \\
\tilde{\mu}_1 &= \mu_1, \quad \tilde{\mu}_2 = \mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1, \\
\tilde{\lambda}_1 &= \lambda, \quad \tilde{\lambda}_2 = \left(1 - \rho\frac{\sigma_2}{\sigma_1}\right)\lambda.
\end{aligned}$$

Lemma

Assume $0 < p < 1,\ A > 0,\ \lambda > 0$ *and* $0 < \rho < \frac{\sigma_1}{\sigma_2}$. *Let* $\tilde{v}$ *be the value function associated with the problem with the new set of futures contracts. Then for any* $(y_1, y_2, x) \in \mathcal{S}_v$ *we have that* $v(y_1, y_2, x) \leq \tilde{v}(\tilde{y}_1, \tilde{y}_2, x)$, *where* $\tilde{y}_2 = y_2,\ \ \tilde{y}_1 = y_1 + \rho\frac{\sigma_2}{\sigma_1}y_2$.

Similarly define: $\tilde{\mathcal{S}}_u = \left\{(\tilde{z}_1, \tilde{z}_2) : 1 - \tilde{\lambda}_1|\tilde{z}_1| - \tilde{\lambda}_2|\tilde{z}_2| > 0\right\}$,

$$\tilde{u}(\tilde{z}_1, \tilde{z}_2) = \tilde{v}(\tilde{z}_1, \tilde{z}_2, 1),\ (\tilde{z}_1, \tilde{z}_1) \in \tilde{\mathcal{S}}_u.$$

Corollary

Under the above assumptions, a viscosity super-solution for $\tilde{u}$ *is an upper bound for* u.

Lower bound

- Auxiliary model not identical to the original model.
- Subsolution.
- Construct on subset of NT^- region and then extend.
- Uses the fact that diffusion is in the radial direction.

Main Theorem 2, when $0 < \rho < \frac{\sigma_1}{\sigma_2}$

Theorem

Assume $0 < p < 1,\ A > 0,\ \lambda > 0,\ \mu_1 > \mu_2,\ \theta_1, \theta_2 > 0$ *and* $0 < \rho < \frac{\sigma_1}{\sigma_2}$. *Fix* $(z_1, z_2) \in \mathcal{S}_u$, *then the value function is*

$$u(z_1, z_2) = \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}} + O(\lambda),$$

where

$$\tilde{\gamma}_2 = A^{p-2} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32} \left(\frac{\tilde{\lambda}_i}{\lambda} \right)^2 (1-p) \tilde{\sigma}_i^2 \tilde{\theta}_i^4 \left(\tilde{\sigma}_1^2 \tilde{\theta}_1^2 + \tilde{\sigma}_2^2 \tilde{\theta}_2^2 \right)^2}.$$

Conclusion

Thank you!

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