Portfolio optimization under partial information with expert opinions

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Introduction

Classical **Merton problem** in dynamic portfolio optimization

- Stock returns $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$

  risk-free interest rate r

- Maximize $E\left[U(X_T)\right]$

  for power utility $U(x) = \frac{x^\theta}{\theta}, \ \theta < 1, \ \theta \neq 0$

- Optimal proportion of wealth invested in risky asset

$$h_t^{(0)} = \frac{1}{1-\theta}\frac{\mu - r}{\sigma^2} = \text{const}$$

  $h^{(0)}$ is a key building block of optimal strategies in more complicated models

Portfolio Optimization and Drift

- Sensitive dependence of investment strategies on **drift** of assets
- Drifts are hard to estimate empirically

  need data over long time horizons
  (other than volatility estimation)
- Problems with stationarity: drift is not constant

Implications

- **Academic literature:** drift is driven by unobservable factors

  Models with partial information, apply filtering techniques

  Björk, Davis, Landén (2010)

  - Linear Gaussian models

    Lakner (1998), Nagai, Peng (2002), Brendle (2006), ...

  - Hidden Markov models

    Sass, Haussmann (2004), Rieder, Bäuerle (2005), Nagai, Rungaldier (2008), Sass, W. (2010),...

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- **Practitioners** use static **Black-Litterman model**

  Apply Bayesian updating to combine

  subjective views (such as “asset 1 will grow by 5%”)
  with empirical or implied drift estimates

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- **Practitioners** use static **Black-Litterman model**

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  subjective views (such as "asset 1 will grow by 5%")
  with empirical or implied drift estimates

- Present paper combines the two approaches

  consider dynamic models with partial observation
  including **expert opinions**

Financial Market Model

$(\Omega, \mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}, P)$ filtered probability space (full information)

Bond $S_t^0 = 1$

Stocks prices $S_t = (S_t^1, \dots, S_t^n)^\top$, returns $dR_t^i = \frac{dS_t^i}{S_t^i}$

$dR_t = \mu(Y_t)\, dt + \sigma\, dW_t$

$\mu(Y_t) \in \mathbb{R}^n$ drift, $\sigma \in \mathbb{R}^{n \times n}$ volatility

W_t n-dimensional $\mathbb{G}$-Brownian motion

Factor process Y_t finite-state Markov chain, independent of W_t

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Factor process Y_t finite-state Markov chain, independent of W_t

state space $\{e_1, \dots, e_d\}$, unit vectors in $\mathbb{R}^d$

states of drift $\mu(Y_t) = MY_t$ where $M = (\mu_1, \dots, \mu_d)$

generator matrix Q

initial distribution $(\pi^1, \dots, \pi^d)^\top$

Investor Information

Investor is not informed about factor process Y_t, he only observes

Stock prices	S_t	or equivalently stock returns R_t
Expert opinions	own view about future performance	
	news, recommendations of analysts or rating agencies	

$\Longrightarrow$ Model with **partial information**.

Investor needs to “learn” the drift from observable quantities.

Expert Opinions

Modelled by marked point process $I = (T_n, Z_n) \sim I(dt, dz)$

- At random points in time $T_n \sim \text{Poi}(\lambda)$ investor observes r.v. $Z_n \in \mathcal{Z}$
- Z_n depends on current state Y_{T_n}, density $f(Y_{T_n}, z)$
  (Z_n) cond. independent given $\mathcal{F}_T^Y = \sigma(Y_s : s \in [0, T])$

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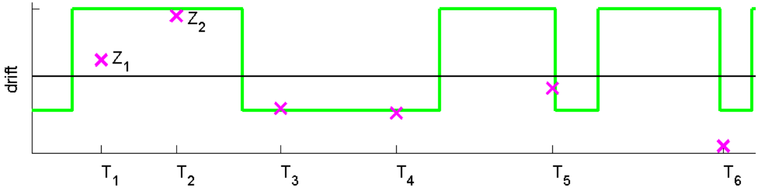
Examples

- Absolute view: $Z_n = \mu(Y_{T_n}) + \sigma_\varepsilon \varepsilon_n$, (ε_n) i.i.d. $N(0,1)$
  The view "S will grow by 5%" is modelled by $Z_n = 0.05$
  σ_ε models confidence of investor



- Relative view (2 assets): $Z_n = \mu_1(Y_{T_n}) - \mu_2(Y_{T_n}) + \widetilde{\sigma}_\varepsilon \varepsilon_n$

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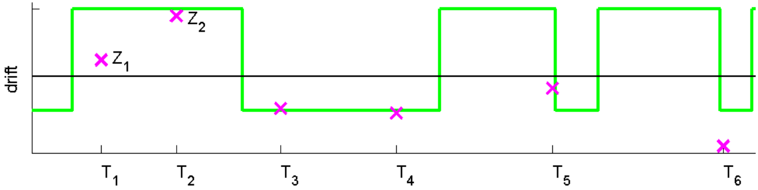

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Investor filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(R_u : u \le t;\ (T_n, Z_n) : T_n \le t)$

Optimization Problem

Admissible Strategies described via portfolio weights $h_t^1, \ldots, h_t^n$

$$\mathcal{H} = \{(h_t)_{t\in[0,T]} \mid h_t \in \mathbb{R}^n, \int_0^T ||h_t||^2 < \infty,$$

$$h \text{ is } \mathbb{F}\text{-adapted} \}$$

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Wealth $dX_t^h = X_t^h h_t^\top \left(\mu(Y_t)\,dt + \sigma dW_t \right), \quad X_0^h = x_0$

Utility function $U(x) = \frac{x^\theta}{\theta}$, power utility, $\theta \in (-\infty, 1) \setminus \{0\}$

$U(x) = \log(x)$ logarithmic utility $(\theta = 0)$

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Value function $V(t,x) = \sup_{h\in\mathcal{H}} v(t,x,h)$

Find optimal strategy $h^* \in \mathcal{H}$ such that $V(0,x_0) = v(0,x_0,h^*)$

Filtering and Reduction to Full Information

HMM Filtering - only return observation

Filter

$$p_t^k := P(Y_t = e_k | \mathcal{F}_t)$$

$$\widehat{\mu(Y_t)} := E[\mu(Y_t)|\mathcal{F}_t] = \mu(p_t) = \sum_{j=1}^{d} p_t^j \mu_j$$

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Innovation process $\widetilde{W}_t := \sigma^{-1}(\, R_t - \int_0^t \widehat{\mu(Y_s)} ds \,)$ is an $\mathbb{F}$-BM

HMM filter Liptser, Shiryaev (1974), Wonham (1965), Elliot (1993)

$$\begin{aligned} p_0^k &= \pi^k \\ dp_t^k &= \sum_{j=1}^{d} Q^{jk} p_t^j dt + a_k(p_t)^\top d\widetilde{W}_t \end{aligned}$$

$$\text{where} \quad a_k(p) \; = \; p^k \sigma^{-1}\Big(\mu_k - \sum_{j=1}^{d} p^j \mu_j\Big)$$

Filtering and Reduction to Full Information (cont.)

HMM Filtering - including expert opinions

Extra information has no impact on filter p_t between ‘information dates’ T_n

Filtering and Reduction to Full Information (cont.)

HMM Filtering - including expert opinions

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Bayesian updating at $t = T_n$:

$$p^k_{T_n} \propto p^k_{T_n-} f(e_k, Z_n) \qquad \text{recall: } f(Y_{T_n}, z) \text{ is density of } Z_n \text{ given } Y_{T_n}$$

$$\text{with normalizer} \quad \sum_{j=1}^{d} p^j_{T_n-} f(e_j, Z_n) =: \bar{f}(p_{T_n-}, Z_n)$$

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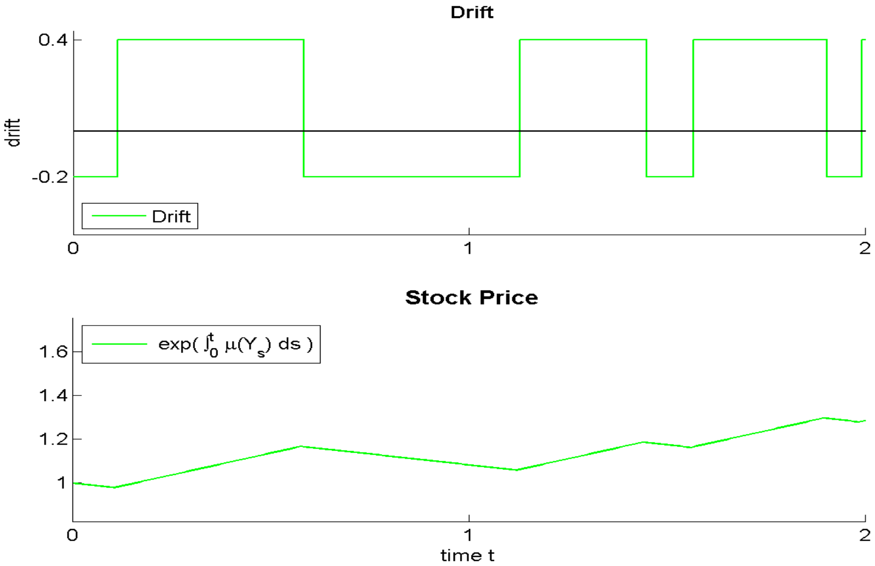
HMM filter

$$\begin{aligned} p^k_0 &= \pi^k \\ dp^k_t &= \sum_{j=1}^{d} Q^{jk} p^j_t dt + a_k(p_t)^\top d\widetilde{W}_t + p^k_{t-} \int_{\mathcal{Z}} \left(\frac{f(e_k, z)}{\bar{f}(p_{t-}, z)} - 1 \right) \gamma(dt \times dz) \end{aligned}$$

Compensated measure $\gamma(dt \times dz) := I(dt \times dz) - \underbrace{\lambda dt \sum_{k=1}^{d} p^k_{t-} f(e_k, z)\, dz}_{\text{compensator}}$

Filtering: Example



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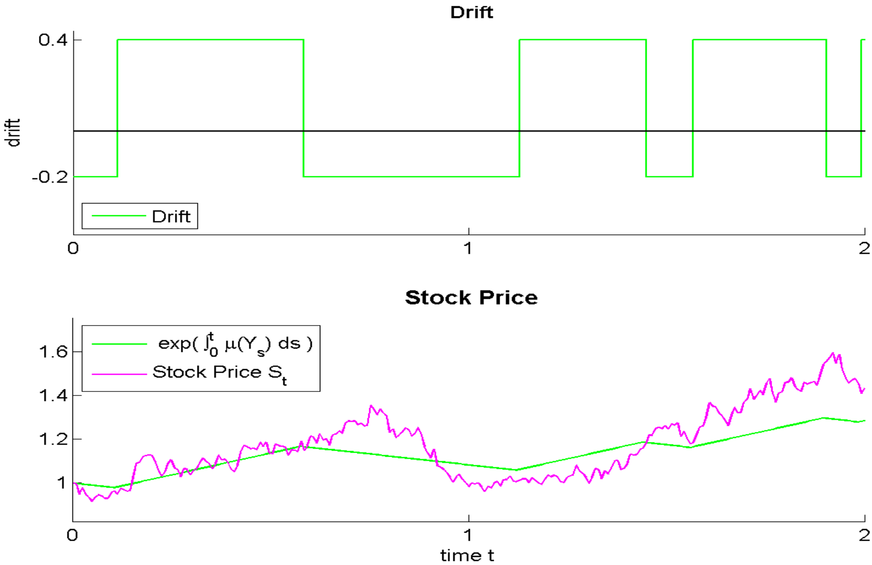

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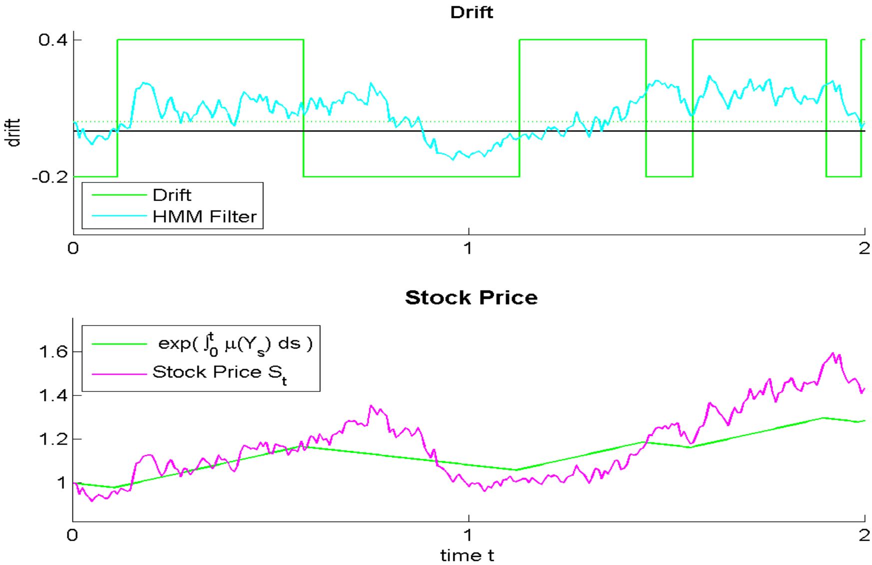

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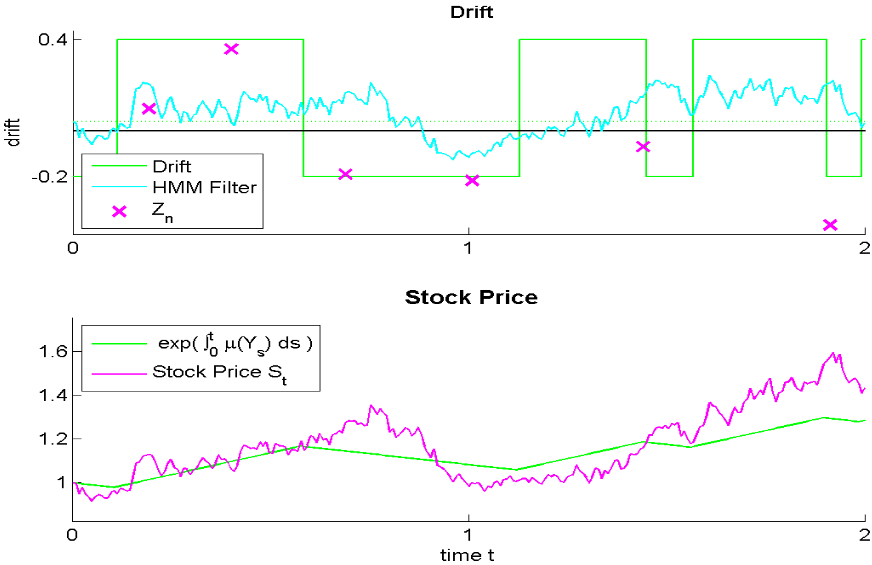

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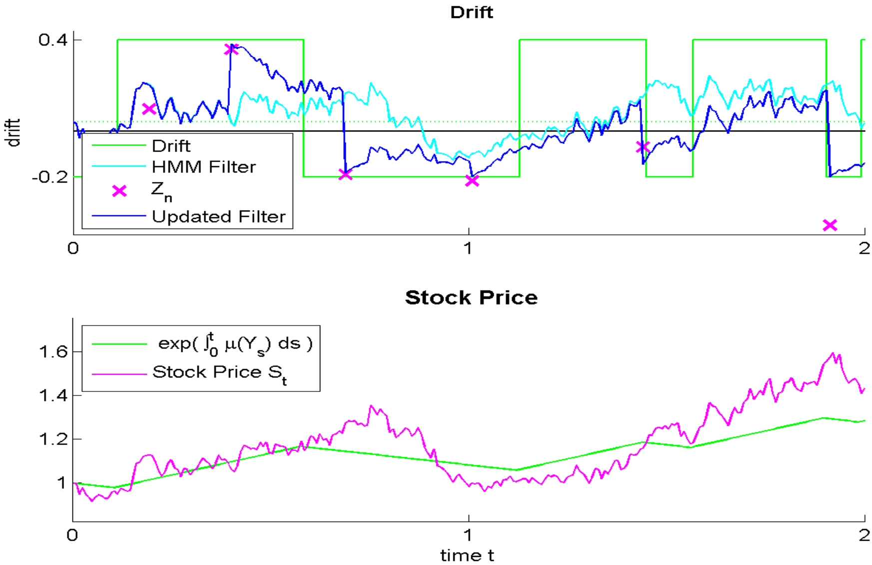

Filtering and Reduction to Full Information (cont.)

Consider augmented state process (X_t, p_t)

Wealth $$dX_t^h = X_t^h \, h_t^\top \, (\underbrace{\widehat{\mu(Y_t)}}_{=M\,p_t} \, dt + \sigma d\widetilde{W}_t), \qquad X_0^h = x_0$$

Filter $$dp_t^k = \sum_{j=1}^{d} Q^{jk} p_t^j dt + a_k(p_t)^\top d\widetilde{W}_t$$

$$+ p_{t-}^k \int_{\mathcal{Z}} \left(\frac{f(e_k, z)}{\overline{f}(p_{t-}, z)} - 1 \right) \gamma(dt \times dz), \qquad p_0^k = \pi^k$$

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Value function $$V(t, x, p) = \sup_{h \in \mathcal{H}} v(t, x, p, h)$$

Find $h^* \in \mathcal{H}(0)$ such that $V(0, x_0, \pi) = v(0, x_0, \pi, h^*)$

Solution for Power Utility

Risk-sensitive control problem (Nagai & Runggaldier (2008))

Let $Z^h := \exp\left\{\theta \int_0^T h_s^\top \sigma d\widetilde{W}_s - \frac{\theta^2}{2} \int_0^T h_s^\top \sigma\sigma^\top h_s ds\right\}$, assume $E[Z^h] = 1$

Change of measure: $P^h(A) = E[Z^h 1_A]$ for $A \in \mathcal{F}_T$

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Reward function

$$E_{t,x,p}[U(X_T^h)] = \frac{x^\theta}{\theta} \underbrace{E_{t,p}^h\left[\exp\left\{-\int_t^T b^{(\theta)}(p_s, h_s)ds\right\}\right]}_{=: v(t,p,h) \text{ independent of } x}$$

$$\text{where} \quad b^{(\theta)}(p,h) := -\theta\left(h^\top Mp - \frac{1-\theta}{2}h^\top \sigma\sigma^\top h\right)$$

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$=: v(t,p,h)$ independent of x

where $b^{(\theta)}(p,h) := -\theta\Big(h^\top Mp - \frac{1-\theta}{2} h^\top \sigma\sigma^\top h\Big)$

Admissible strategies $\mathcal{A} = \mathcal{H} \cap \{ (h_t) \mid E[Z^h] = 1\}$

Value function $V(t,p) = \sup_{h\in\mathcal{A}} v(t,p,h)$

Find $h^* \in \mathcal{A}$ such that $V(0,\pi) = v(0,\pi,h^*)$

HJB-Equation

$$V_t(t,p) + \sup_{h\in\mathbb{R}^n} \left\{ \mathcal{L}^h V(t,p) - b^{(\theta)}(p,h) V(t,p) \right\} = 0$$

$$\text{terminal condition} \quad V(T,p) = 1$$

where $\mathcal{L}^h$ generator of the filter process p_t under measure P^h

HJB-Equation

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Optimal Strategy

$$
h^* = h^*(t,p) = \frac{1}{(1-\theta)} (\sigma\sigma^\top)^{-1} \left\{ Mp + \frac{1}{V(t,p)} \sigma \sum_{k=1}^{d} a_k(p) V_{p^k}(t,p) \right\}
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myopic strategy $+$ correction

Certainty equivalence principle does not hold

HJB-Equation (cont.)

Plugging in h^* into the HJB equation and substituting $V = G^{1-\theta}$ we derive a

Transformed HJB-Equation for $G = G(t,p)$

$$G_t + \frac{1}{2} tr[A^\top(p) A(p) D^2 G] + B^\top(p) \nabla G + C(p) G$$

$$+ \frac{\lambda}{1-\theta} \int_{\mathcal{Z}} \frac{G^{1-\theta}(t, p + \Delta(p,z)) - G^{1-\theta}(t,p)}{G^{-\theta}(t,p)} \bar{f}(p,z) dz = 0,$$

$$G(T,p) = 1,$$

The functions A, B, C, Δ are defined in the paper.

Note that the equation has a **linear diffusion part** but **nonlinear integral term**.

Policy Improvement

Starting approximation is the myopic strategy $h_t^{(0)} = \frac{1}{1-\theta}(\sigma\sigma^\top)^{-1}Mp_t$

The corresponding reward function is

$$V^{(0)}(t,p) := v(t,p,h^{(0)}) = E_{t,p}\Big[\exp\Big(-\int_t^T b^{(\theta)}(p_s^{h^{(0)}}, h_s^{(0)})ds\Big)\Big]$$

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Consider the following optimization problem

$$\max_h \left\{\mathcal{L}^h V^{(0)}(t,p) - b^{(\theta)}(p,h)V^{(0)}(t,p)\right\}$$

with the maximizer

$$h^{(1)}(t,p) = h^{(0)}(t,p) + \frac{1}{(1-\theta)V^{(0)}(t,p)}(\sigma^\top)^{-1}\sum_{k=1}^{d} a_k(p)\, V_{p^k}^{(0)}(t,p)$$

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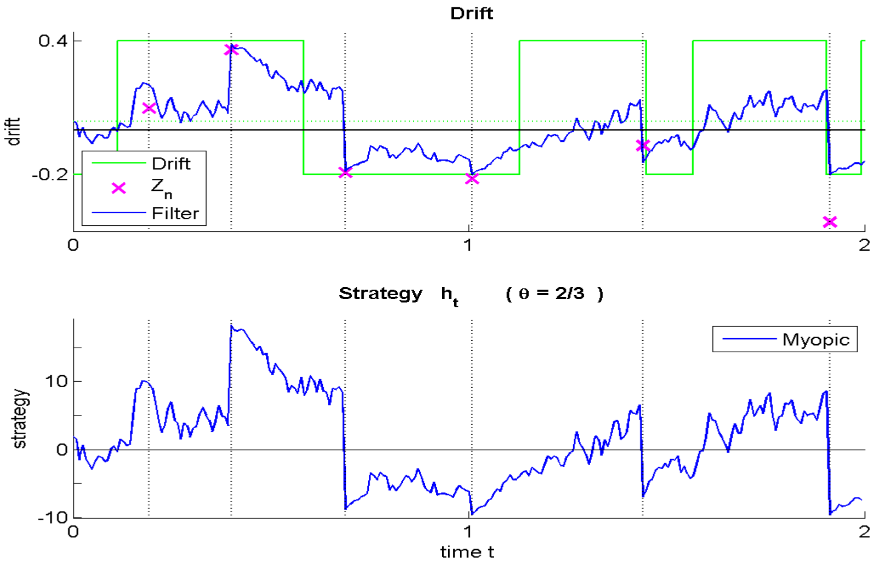
For the corresponding reward function $V^{(1)}(t,p) := v(t,p,h^{(1)})$ it holds

Lemma ($h^{(1)}$ is an improvement of $h^{(0)}$)

$$V^{(1)}(t,p) \geq V^{(0)}(t,p)$$

Numerical Results



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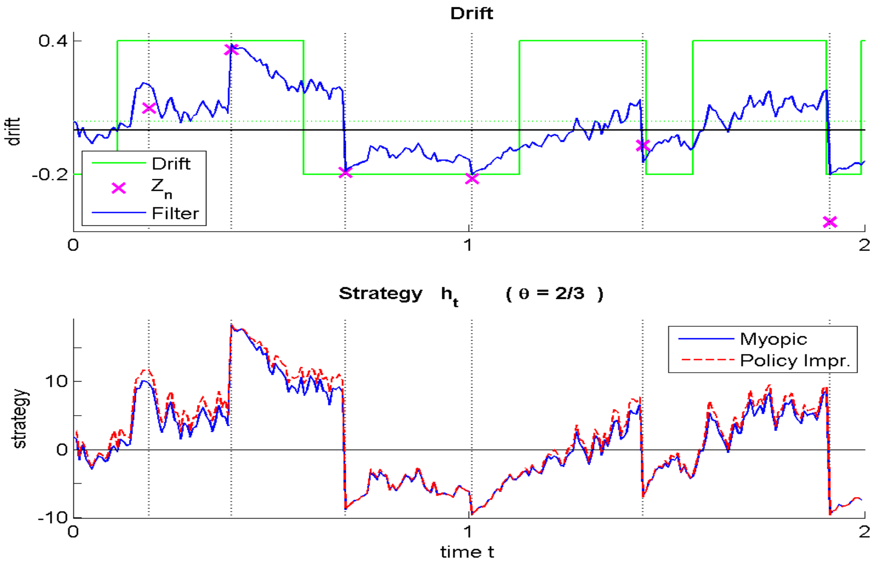

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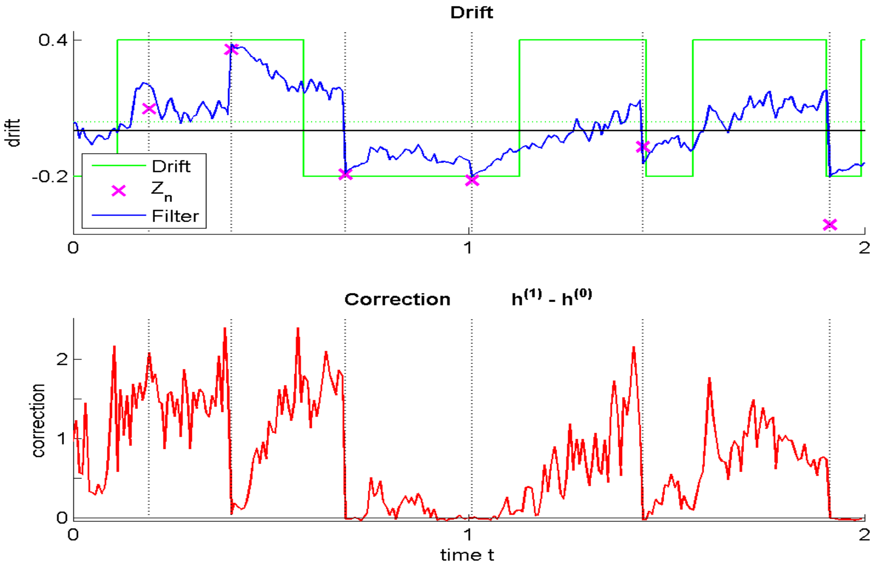

For $t = T_n$: nearly full information $\Longrightarrow$ correction ≈ 0