MULTIVARIATE EXTENSION OF PUT-CALL SYMMETRY

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Barrier-contingent claims

- $S_t = (S_{01} e^{t\lambda_1} e^{\xi_{t1}}, \dots, S_{0n} e^{t\lambda_n} e^{\xi_{tn}}), t \in [0, T]$
  ($\lambda_1, \dots, \lambda_n$ — deterministic carrying costs)
- $S_T = F\eta = (F_1\eta_1, \dots, F_n\eta_n)$

Barrier-contingent claim:

$$X = f(S_T)\mathbb{I}_{\{\dots\}} = f(F\eta)\mathbb{I}_{\{\dots\}}$$

where $\mathbb{I}_{\{\dots\}}$ is the indicator of some barrier event and f is some payoff function, e.g. ($k > 0$)

$$f(S_T) = (w_1F_1\eta_1 + \cdots + w_nF_n\eta_n - k)_+ \,,$$

$$f(S_T) = (\max(w_1F_1\eta_1, \dots, w_nF_n\eta_n) - k)_+ \,,$$

$$f(S_T) = (w_1F_1\eta_1 + \cdots + w_nF_n\eta_n)_+ \,.$$

Symmetries

Well-known classic European put-call symmetry (holding for *certain* models)

$$\mathbf{E}(F\eta - k)_+ = \mathbf{E}(F - k\eta)_+ \quad \text{for every } k \geq 0\,.$$

In view of that, consider

- $\eta = (\eta_1, \dots, \eta_n)$, $(1, \eta_1, \dots, \eta_n)$ — random price changes
- $f(\eta)$ — payoff function (forward prices are included in the payoff functions)
- Discussion: In which case is $\mathbf{E}_\mathbf{Q} f(\eta)$ invariant with respect to swaps of its arguments (expectation w.r.t. martingale measure)?

Main application: *Semi-static hedging* of certain barrier-contingent claims, i.e. the replication of these contracts by trading European-style claims at no more than two times after inception.

Some historic remarks

- *Bates' rule*:

  D.S. Bates. In particular: The skewness premium. Adv. Fut. Options Res., 1997; see also J. Bowie and P. Carr, Static simplicity, Risk, 1994.

- *Semi-static hedge* of barrier options (based on J. Bowie and P. Carr):

  (a call option at the barrier can be converted in certain put options)

  P. Carr, K. Ellis, V. Gupta. J. Finance, 1998; P. Carr, R. Lee, 2009.

- *Lévy markets*:

  J. Fajardo, E. Mordecki. Symmetry and duality. Quant. Finan., 2006.

- *Multiasset* case:

  I. M., M. S., 2010.

Duality principle alone does not suffice

For the duality principle, see Eberlein, Papapantoleon & Shiryaev 2008, 2009 and the literature cited therein.

Since $\mathbf{E}_{\mathbf{Q}}\eta = 1$, define

$$\frac{d\tilde{\mathbf{Q}}}{d\mathbf{Q}} = \eta\,.$$

With $\tilde{\eta} = \eta^{-1}$

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}(H\eta - k)_+ &= \mathbf{E}_{\tilde{\mathbf{Q}}}\eta^{-1}(H\eta - k)_+ = \mathbf{E}_{\tilde{\mathbf{Q}}}(H - k\tilde{\eta})_+ \\ &= kH^{-1}\mathbf{E}_{\tilde{\mathbf{Q}}}(H^2k^{-1} - H\tilde{\eta})_+\,.\end{aligned}$$

Need

$$\mathbf{E}_{\mathbf{Q}}(H\eta - k)_+ = kH^{-1}\mathbf{E}_{\mathbf{Q}}(H^2k^{-1} - H\eta)_+$$

(resp. equivalent properties) for symmetry based semi-static hedges.

Most important multivariate functions

- Basket option $\mathbf{E}_{\mathbf{Q}}\left(u_0 + u_1\eta_1 + \cdots + u_n\eta_n\right)_+$ function of $(\eta_0 = 1, \eta_1, \ldots, \eta_n)$

- Calls (puts) on maximum/minimum, e.g.

$$\mathbf{E}_{\mathbf{Q}}(\max(u_1\eta_1, \ldots, u_n\eta_n) - u_0)_+$$

  for our symmetry analysis can be replaced by $\mathbf{E}_{\mathbf{Q}} \max(u_0, u_1\eta_1, \ldots, u_n\eta_n)$

- Exchange option $\mathbf{E}_{\mathbf{Q}}\left(u_1\eta_1 + \cdots + u_n\eta_n\right)_+$

Characterisation of distributions

- Breeden & Litzenberger (1978): the prices of all call (resp. put) options determine the distribution of the single underlying.
- The prices of all basket options determine the multiasset distribution
  Carr & Laurence — absolutely continuous case;
  the general case is implicit in Henkin & Shananin, Koshevoy & Mosler.

- The same holds for all options on the maximum (weighted) $\max(u_0, u_1\eta_1, \ldots, u_n\eta_n)$ or minimum $\min(u_0, u_1\eta_1, \ldots, u_n\eta_n)$.
- The same holds for calls (puts) on maximum/minimum, e.g.

$$(\min(u_1\eta_1, \ldots, u_n\eta_n) - u_0)_+ \,.$$

Does not hold for exchange options $(u_1\eta_1 + \cdots + u_n\eta_n)_+$.

Information in exchange options

Let $\eta = e^{\xi}$ and $\eta^* = e^{\xi^*}$ be integrable random vectors. Then

$$\mathbf{E}(\langle u, \eta\rangle)_+ = \mathbf{E}(\langle u, \eta^*\rangle)_+ \quad \text{for all } u \in \mathbb{R}^n$$

if and only if

$$\varphi_{\xi}(u - \boldsymbol{\imath} w) = \varphi_{\xi^*}(u - \boldsymbol{\imath} w) \qquad (1)$$

for all $u \in \mathbb{H}$, where

$$\mathbb{H} = \{u \in \mathbb{R}^n : \sum_{k=1}^{n} u_k = 0\},$$

and for at least one (and then necessarily for all) w, such that $\sum w_i = 1$ and both sides in (1) are finite.

Infinitely divisible case: (1) can be expressed via the Lévy triplet.

Consequences

- Prices of all basket options determine the prices of all European options (depending on the same assets, with the same maturity).
- Prices of all exchange options determine them for a *certain* class of payoff functions.

Symmetries of multivariate option prices functions

- Basket option $\mathbf{E_Q}\left(u_0 + u_1\eta_1 + \cdots + u_n\eta_n\right)_+$
  (swap u_0 and u_i) — η is i-self-dual (for all $(u_0, u) \in \mathbb{R}^{n+1}$)
- Option on the maximum $\mathbf{E_Q}\max(u_0, u_1\eta_1, \ldots, u_n\eta_n)$
  (swap u_0 and u_i) — η is i-self-dual (for all $(u_0, u) \in \mathbb{R}^{n+1}$)
- Exchange option $\mathbf{E_Q}\left(u_1\eta_1 + \cdots + u_n\eta_n\right)_+$
  (swap u_i and u_j with $u_0 = 0$) — η is ij-swap-invariant
  (for all $u \in \mathbb{R}^n$)

Characterisation of self-dual distributions

Integrable η is i-self-dual if and only if e.g.

- $\mathbf{E}f(\eta) = \mathbf{E}[f(\varkappa_i(\eta))\eta_i]$ for all integrable payoffs f, where

$$\varkappa_i(x) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{1}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) .$$

- The distribution of η under $\mathbf{Q}$ coincides with the distribution of $\varkappa_i(\eta)$ under $\mathbf{Q}^i$, where

$$\frac{d\mathbf{Q}^i}{d\mathbf{Q}} = \eta_i .$$

- If η is absolutely continuous, $p_\eta(x) = x_i^{-n-2} p_\eta(\varkappa_i(x))$ a.e.

- Characterisation in terms of the distribution of $\xi = \log \eta$

$$\varphi_\xi\left(u - \frac{1}{2}\,\boldsymbol{\imath} e_i\right) = \varphi_\xi\left(K_i^\top u - \frac{1}{2}\,\boldsymbol{\imath} e_i\right), \quad u \in \mathbb{R}^n\,,$$

where

$K_i x = (x_1 - x_i, \ldots, x_{i-1} - x_i, -x_i, x_{i+1} - x_i, \ldots, x_n - x_i)$,

(some other equivalent complex shifts are also possible).

Infinitely divisible case: This characterisation can be expressed via the Lévy triplet.

PCS in the one asset case

- Classic European put-call symmetry is equivalent to many other definitions.
- Almost any tail behaviour is possible.
- η has a non-negative skewness and for infinitely divisible $\xi = \log \eta$, ξ has non-positive skewness.
- For much more, see Carr and Lee 2009 and the literature cited therein.

Swap-invariance and PCS

Integrable η is called ij*-swap-invariant* if

$$\mathbf{E}_{\mathbf{Q}}(u_1\eta_1 + \cdots + u_n\eta_n)_+ \,, \quad u \in \mathbb{R}^n \,,$$

is π_{ij}-invariant (swap u_i and u_j).

Integrable η is ij*-swap-invariant* if and only if the $(n-1)$-dimensional random vector

$$\tilde{\varkappa}_j(\eta) = \left(\frac{\eta_1}{\eta_j}, \dots, \frac{\eta_{j-1}}{\eta_j}, \frac{\eta_{j+1}}{\eta_j}, \dots, \frac{\eta_n}{\eta_j}\right)$$

is *self-dual* with respect to the ith component under $\mathbf{Q}^j$.

Characterisation

An integrable random vector $\eta = e^{\xi}$ is ij-swap-invariant if and only if the characteristic function of ξ satisfies

$$\varphi_\xi(u - \boldsymbol{\imath}\frac{1}{2}\, e_{ij}) = \varphi_\xi(\pi_{ij}u - \boldsymbol{\imath}\frac{1}{2}\, e_{ij})$$

for all

$$u \in \mathbb{H} = \{u \in \mathbb{R}^n :\ \sum_{k=1}^{n} u_k = 0\}\,,$$

where $e_{ij} = e_i + e_j$ (many equivalent complex shifts).

Infinitely divisible case: This characterisation can be expressed via the Lévy triplet.

Examples

- Black-Scholes case: *Each bivariate* risk-neutral log-normal distribution is swap-invariant, no matter what volatilities of the assets and correlation are.
- The considerable effective degrees of freedom for modelling two assets based on dependent generalised hyperbolic Lévy processes only slightly decrease if we ensure that the bivariate swap-invariance property holds.
- Etc.

Example: Certain knock-out Margrabe ($n = 2$)

- Payoff

$$X_{\text{sw}} = (S_{T1} - S_{T2})_+ \mathbb{1}_{c > \frac{S_{t2}}{S_{t1}} \; \forall t \in [0,T]}$$

with $c \geq 1$, $0 < \frac{S_{02}}{S_{01}} < c$, and (for simplicity) assume $(S_{t1}, S_{t2}) = (S_{01} e^{\lambda t} e^{\xi_{t1}}, S_{02} e^{\lambda t} e^{\xi_{t2}})$, (ξ_{t1}, ξ_{t2}), $t \in [0, T]$, is a Brownian motion with drift and non singular covariance matrix

$$\mu = -\left(\frac{\sigma_1^2}{2}, \frac{\sigma_2^2}{2}\right) \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} .$$

- Hedge portfolio:
  - long position in the Margrabe option with payoff function $(S_{T1} - S_{T2})_+$,
  - short position in the weighted Margrabe option with payoff function $(c^{-1} S_{T2} - c S_{T1})_+$.

Verification of the hedge

- If the barrier is not hit, then $cS_{t1} > S_{t2}$ for all t; the short position $(c^{-1}S_{T2} - cS_{T1})_+$ expires worthless and the long position $(S_{T1} - S_{T2})_+$ replicates the option.
- If $cS_{\tau 1} = S_{\tau 2}$, then the values of these two options at time τ are identical.

Problems with carrying costs

Write

$$e^{\lambda}\eta = \left(e^{\lambda_1+\xi_1}, \ldots, e^{\lambda_n+\xi_n}\right),$$

where $\lambda_i = r - q_i$ (q_i-dividend yield), $i = 1, \ldots, n$ (and for simplicity of notation $T = 1$).

The problem in self-dual cases

- For applications usually $\mathbf{E}e^{\xi_j} = 1, j = 1, \ldots, n$.
- Multiplication by e^{λ_i}, $\lambda_i \neq 0$, moves the expectation away from one.
- $e^{\lambda+\xi}$ self-dual with respect to the ith coordinate $\Rightarrow \mathbf{E}e^{\lambda_i+\xi_i} = 1$.
- For semi-static hedging, symmetry is rather needed in $e^{\lambda+\xi}$ than in e^{ξ}.

Quasi-self-duality

$\eta = e^{\xi}$ is *quasi-self-dual* (with respect to the ith coordinate) if there exist $\lambda \in \mathbb{R}^n$ and $\alpha \neq 0$ such that $(e^{\lambda+\xi})^{\alpha}$ is integrable and self-dual with respect the ith coordinate.

Univariate power-transform: Carr and Lee (2009), based on earlier work of Carr and Chou.

For the multivariate case

$$\mathbf{E} f(S_T)$$
$$= \mathbf{E}\Big[f\Big(\frac{S_{0i}}{S_{Ti}} (S_{T1}, \ldots, S_{T(i-1)}, S_{0i}, S_{T(i+1)}, \ldots, S_{Tn}) \Big) \Big(\frac{S_{Ti}}{S_{0i}} \Big)^{\alpha} \Big],$$

etc.

A similar extension to quasi-swap-invariance is known (useful for non-equal carrying costs).

Finding α in infinitely divisible cases

To ensure that $\mathbf{E}\eta_i = 1$ the value α must satisfy

$$a_{ii}\alpha = a_{ii} - 2\lambda_i + 2\int_{\mathbb{R}^n} (e^{x_i} - 1 - x_i e^{\frac{\alpha}{2}x_i} \mathbb{I}_{|||x|||\le 1}) d\nu(x)\,,$$

where $|||x|||^2 = \frac{1}{2}\left(\|x\|^2 + \|K_i x\|^2\right)$.

Usually not easy to solve (even for $n = 1$) and solution(s) may not exist.

There are some friendly special cases.

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