Monsoon options: early-exercise Asian tail using fast & accurate hybrid numerical techniques

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Outline

- introduction
- contract space
- motivation & the modelling scenario
- commodity futures model
- geometric averaging
  - Asian tail
  - Monsoon
- arithmetic averaging
  - Asian tail
  - Monsoon
- results

Introduction to the contract

We introduce and value a class of option that is a surprising omission from the current literature, which we christen the ‘Monsoon’ option.

- The number of exercise opportunities, I. Unconventionally, the corresponding time to each opportunity does not necessarily equate to the termination of the contract, rather it is simply when a decision may be taken.
- The averaging period for Asian-style options, $\bar{T}$. Best expressed as a fraction of the option’s life, $\bar{T}/T$.

Contract	$\bar{T}/T$	I	notes
Monsoon	$[0,1]$	$[1,\infty)$	
- Vanilla Asian	1	$[1,\infty)$	trivial when $I \neq 1$
- Asian tail or forward-starting Asian	$[0,1]$	1	
- European	0	1	
- Bermudan	0	$[1,\infty)$	
- American	0	∞	

(Law 2009, Bilger 2004)

Contract space

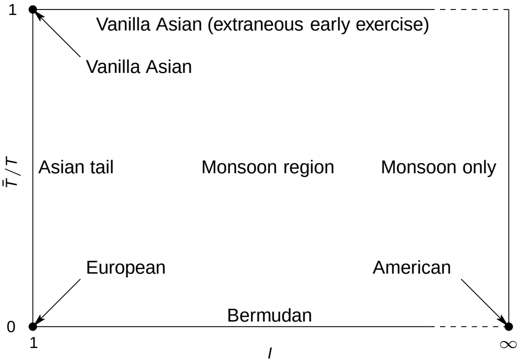

Figure: Schematic of the contract space encompassed by a Monsoon option.

Motivation and modelling scenario

The Monsoon contract encompasses several notable features

- Asian features
  - less susceptible to market manipulation
  - reduce contract volatility - generally cheaper than European
- Early-exercise
  - desirable but expensive

Leads us initially to commodities

- consumed on a continual basis - Asian
- choice of delivery dates - early-exercise

Also note

- Not to be confused with Hawaiian options (Jørgensen, et al. 1999) - conflicting features
- Bars the use of perpetuals (Chung & Shackleton 2007)

Commodity futures model

Basic model with spot commodity S & convenience yield y (Schwartz 1997)

$$\frac{\mathrm{d}S}{S} = (r - y)\,\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}$$

$$\mathrm{d}y = (\hat{y} - \kappa_y y)\,\mathrm{d}t + \sigma_y \mathrm{d}W_y^{\mathbb{Q}}$$

Leads to futures contract $F(t; T_F)$ maturing at T_F as the underlying (Hilliard & Reis 1998)

$$\frac{\mathrm{d}F}{F} = \sigma(t)\mathrm{d}W_F^{\mathbb{Q}}$$

Time-dependent volatility

$$\sigma(t; T_F) = \sqrt{\sigma_S^2 + \sigma_y^2 B^2 + 2\rho_{Sy}\sigma_S\sigma_y B}, \quad B(t; T_F) = \frac{1}{\kappa_y}\left(e^{-\kappa_y (T_F - t)} - 1\right)$$

- We ignore stochastic interest rate.
- Developed further with no-arbitrage model (Trolle & Schwartz 2009).

Numerical approach: European example

Four pricing problems, all satisfying Monsoon option criteria:

1. Geometric: Asian tail ... QUAD & Analytic
2. Geometric: Monsoon ... early-exercise QUAD & Analytic
3. Arithmetic: Asian tail ... QUAD & finite-difference
4. Arithmetic: Monsoon ... early-exercise QUAD & finite-difference

QUAD: European example

In order to introduce the notation and valuation approach, begin with simplest option contract

$$V^{\mathrm{E}}(F,T;T)=\begin{cases} V_0^{\mathrm{CE}}=[F(T)-K]^+ \\ V_0^{\mathrm{PE}}=[K-F(T)]^+ \end{cases}$$

Standard hedging arguments with futures (Black 1976) yield

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2F^2\frac{\partial^2 V}{\partial F^2}-rV=0$$

Simply solved by transformations to the heat equation

$$V^{\mathrm{E}}(x,t;T)=\frac{e^{-r(T-t)}}{2\sqrt{\pi v}}\int_{-\infty}^{\infty}V_0^{\mathrm{E}}(x')\exp\left[-\frac{(x-x')^2}{4v}\right]\mathrm{d}x'$$

We solve using QUAD (Andricopoulos, et al. 2003), with substitution of limits

$$\hat{x}(F,t_i,t_{i+1})=x(F,t_i)+D\left(\int_{t_i}^{t_{i+1}}\sigma^2(t')\mathrm{d}t'\right)^{1/2}$$

QUAD: Numerical integration specific to derivatives.

Geometric options: Asian tail

Geometric options: Asian tail: pricing method

Geometrics are easiest to value, so good starting point

$$V^{\mathrm{GT}}(\bar{F}, T; t^*, T) = \begin{cases} V_0^{\mathrm{CGT}} = [\bar{F}^{\mathrm{GT}}(t^*, T) - K]^+ \\ V_0^{\mathrm{PGT}} = [K - \bar{F}^{\mathrm{GT}}(t^*, T)]^+ \end{cases}$$

Start of averaging period t^*; end of averaging and maturity of the contract T. Note that we use $\bar{T} = T - t^*$.
The geometric average of the tail $\bar{F}^{\mathrm{GT}}(t^*, T)$ given by

$$\bar{F}^{\mathrm{GT}}(t^*, t) = \exp\left[\frac{1}{t - t^*}\int_{t^*}^{t} \ln(F(t'))\mathrm{d}t'\right] \text{ for } t \geq t^*$$

Solution method - split problem into two regions

- Tail region $t \in [t^*, T]$, analytic evaluation of the Asian option
- 'European' region $t \in [t_0, t^*]$, solved using QUAD, with 'payoff' given by the tail prices at nodes at t^*

Geometric options: Asian tail: pricing schematic

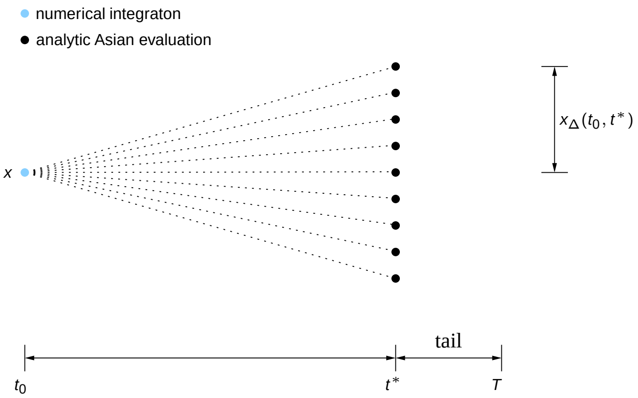

Figure: Schematic of Asian tail QUAD. Nodes are highlighted to indicate how they are evaluated.

Geometric options: Asian tail: The tail region

Using a convenient definition (Kemna & Vorst 1990)

$$G(t) = \int_{t^*}^{t} \ln\left(F(t')\right) \mathrm{d}t', \quad G \in (-\infty, \infty)$$

We arrive at the PDE for the fixed strike Asian option price process

$$\frac{\partial V}{\partial t} + \ln(F)\frac{\partial V}{\partial G} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0 \text{ for } t^* \leq t \leq T$$

Which prices options at t^* over values of $F \leftrightarrow x$, i.e. the put option

$$V^{\mathrm{PGR}}(F, t^*; \bar{T}) = e^{-r\bar{T}} \left\{ K\Phi\left(-d_2\right) - Fe^{u-w}\Phi\left(-d_1\right) \right\}$$

respectively, where $\Phi(d)$ is the cumulative normal distribution function.

Geometric options: Asian tail: The European region

Identical to the pure European pricing option, with maturity t^*, where the payoff is given by the tail.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0 \text{ for } t_0 \leq t \leq t^*$$

With the payoff

$$V_0^{\mathrm{GR}}(x) = V^{\mathrm{GR}}(F, t^*),$$

And thus we can use QUAD with our modified limits of integration to yield

$$V^{\mathrm{GT}}(x, t_0; t^*) = \frac{e^{-r(t^*-t_0)}}{2\sqrt{\pi v}} \int_{\check{x}(t_0,t^*)}^{\hat{x}(t_0,t^*)} V_0^{\mathrm{GR}}(x') \exp\left[-\frac{(x-x')^2}{4v}\right] \mathrm{d}x'$$

Geometric options: Monsoon

Geometric options: Monsoon: The tail region

Early-exercise Asian tail - where 'exercise' triggers the commencement of the averaging process

$$V(\bar{F}, t; \bar{T}, T) = \begin{cases} V^{\text{CGM}} \geq [\bar{F}^{\text{GT}}(t, t+\bar{T}) - K]^+ \\ V^{\text{PGM}} \geq [K - \bar{F}^{\text{GT}}(t, t+\bar{T})]^+ \end{cases}$$

Given early-exercise, we must locate the free-boundary, so the payoff is given by

$$V_0^{\text{GM}*}(x, t_i) = V_0^{\text{GR}}(x) \text{ if } i = I,$$
$$V_0^{\text{GM}*}(x, t_i) = \left[V_0^{\text{GR}}(x), V^{\text{GM}}(x, t_i)\right]^+ \text{ if } i < I.$$

The Monsoon contract price is given by

$$V^{\text{GM}}(x, t_i; t_{i+1}) = \frac{e^{-r(t_{i+1}-t_i)}}{2\sqrt{\pi v}} \int_{\check{x}(t_i,t_{i+1})}^{\hat{x}(t_i,t_{i+1})} V_0^{\text{GM}*}(x', t_{i+1}) \exp\left[-\frac{(x-x')^2}{4v}\right] \mathrm{d}x'$$

- We can curtail local integration to save computing time.
- This describes Bermudan implementation. Richardson extrapolation for Americans. (Andricopoulos et al. 2003)

Geometric options: Monsoon schematic

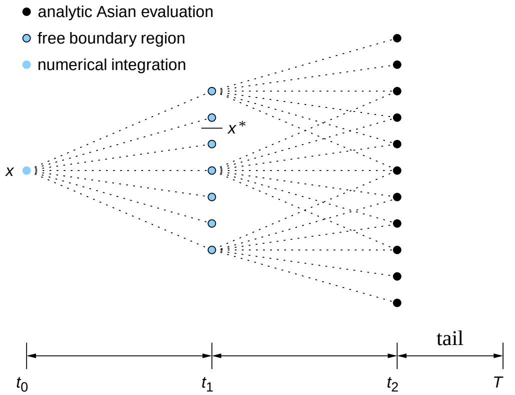

Figure: Schematic of two-step geometric Monsoon QUAD. Nodes are highlighted to indicate how they are evaluated.

Arithmetic options: Asian tail

Arithmetic options: Asian tail I

We begin by stating the continuous arithmetic average

$$\bar{F}^{\mathrm{A}}(t) = \frac{1}{t - t^*} \int_{t^*}^{t} F(t')\mathrm{d}t' \text{ for } t \geq t^*$$

Neat method to reduce the dimensionality of the problem (Večeř 2001, Večeř 2002). The identity $\mathrm{d}(tF) = t\mathrm{d}F + F\mathrm{d}t$ permits

$$\bar{F}^{\mathrm{A}}(t) = F(t^*) + \int_{F(t^*)}^{F(t)} \left(1 - \frac{t'}{t - t^*}\right) \mathrm{d}F(t')$$

So we can express the option in terms of a 'traded account' X, rather than average $\bar{F}$, so $V(F, X, t) = V(F, \bar{F}, t)$, where the account follows

$$X(t) = X(t^*) + \int_{F(t^*)}^{F(t)} q(t')\mathrm{d}F(t')$$

given a holding strategy $q(t)$.

Arithmetic options: Asian tail II

We obtain the PDE for the process on the traded account

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}(\sigma F)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(q\sigma F)^2 + \frac{\partial^2 V}{\partial F \partial X} q(\sigma F)^2 - rV = 0$$

Substitutions $z(X, F) = \frac{X}{F}$ and $U(z, t) = \frac{V}{F}$ turn the problem from 3D to 2D

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial z^2}\sigma^2 (z - q)^2 - rU = 0$$

With the boundary conditions

$$\frac{\partial U}{\partial t} - rU \to 0 \quad \text{as} \quad z \to +\infty; \qquad \frac{\partial U}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty$$

Solve by

- Use of 2D finite-difference grid to price the Asian (tail) region from T to t^*.
- Grid can value options (payoffs) at t^* over range of underlying F in one go.
- Map from grid to QUAD $U(z) \to V(x)$. Use polynomial interpolation.

Arithmetic options: Asian tail schematic

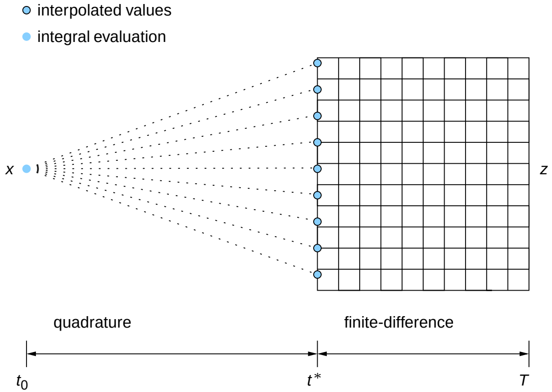

Figure: Schematic of arithmetic Asian tail QUAD-finite-difference numerical scheme. Nodes are highlighted to show how they are evaluated.

Arithmetic options: Monsoon

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To extend the Arithmetic method to pricing early-exercise options

- Identical treatment of 'European' region, and general method as geometric Monsoon.
- We solve one finite-difference grid per exercise time.
- Locating the free boundary will then only perform polynomial interpolation on the grid, not recalculating.

Results overview

General overview of results is best illustrated by some figures, but

- General behaviours
  - Even with short tails, multiple-exercise Monsoons are considerably cheaper than Bermudan equivalent, and often Europeans.
- Arithmetic vs. Geometric
  - Geometric average is always smaller than arithmetic (Levy 1992): arithmetic $\neq$ geometric.
  - Given short tails and/or low volatility, geometric method is a practical substitute for Arithmetic.
  - Calculation times for geometric Monsoon options are a matter of seconds, arithmetics less than a minute on modern systems.

Following figures are for contracts with:

- strike $K = 100$
- maturity $T = 3$ months
- absolute price differences shown

Illustration of contract prices

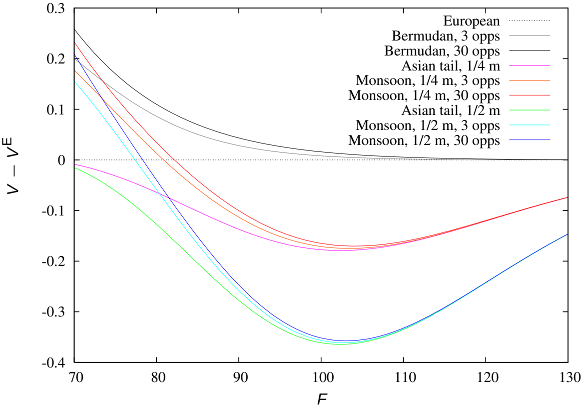

Figure: Geometric put option prices at initiation relative to European. $K = 100$, $T = 0.25$. QUAD numerics: Simpsons, $D = 12$, $x_\delta = 10^{-3}$, $\varepsilon = 10^{-9}$.

Difference between arithmetic and geometric averaging

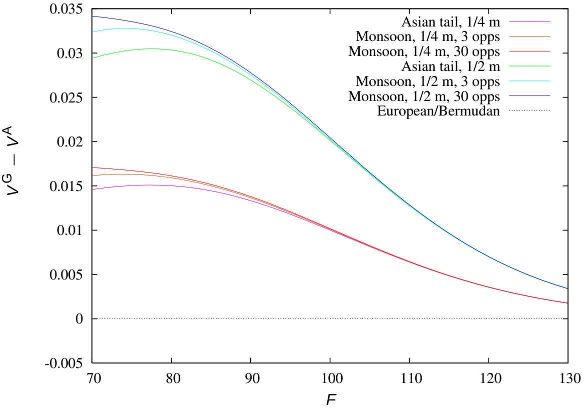

Figure: Absolute price premium of geometric put option prices over arithmetic at initiation. QUAD numerics: Simpsons, $D = 12$, $x_\delta = 10^{-3}$, $\varepsilon = 10^{-9}$. FD numerics: $t_\delta = 5 \times 10^{-4}$, $z_\delta = 5 \times 10^{-4}$, $\hat{z} = 2$, $\check{z} = -2$. Third-order polynomial interpolation.

End & contact

Thank you for listening

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