Maximum Drawdown Insurance

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   - Insurance against Drawing Down
2. **(Semi-)Robust Replication of Digital Call on Maximum Drawdown**
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   - Hedging with Single Barrier One Touches
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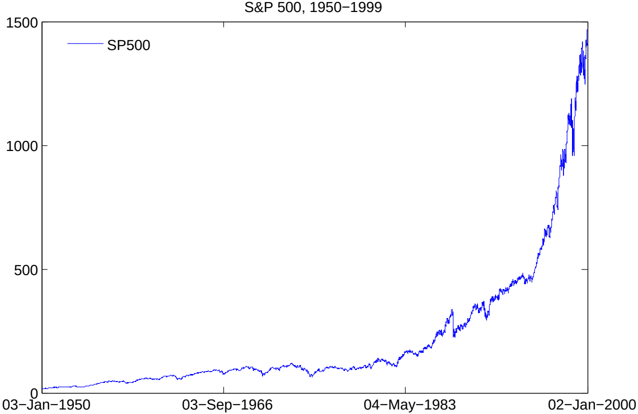
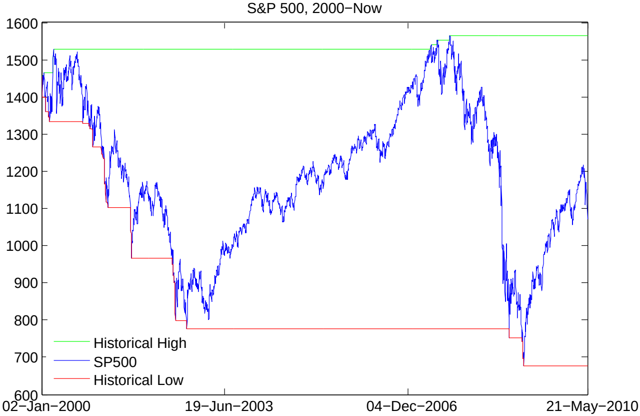
Drawdowns and Drawups

- In the long term, the price of a stock or index is growing. This suggests the so-called “buy and hold” strategy.



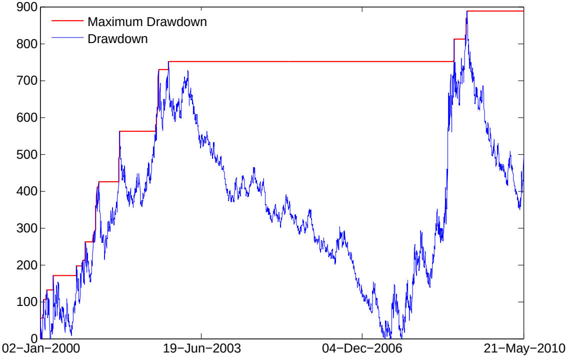
Drawdowns and Drawups

- In the short term, the price of a stock or index is fluctuating, and may have a big drop or a big rally over a period $[0, T]$.
    - The present decrease from the historical high $D_t = \sup_{s\in[0,t]} F_s - F_t$.
    - The present increase over the historical low $U_t = F_t - \inf_{s\in[0,t]} F_s$.



Maximum Drawdown and Drawup, and Their Derivatives

- Maximum drawdown $MD_T = \sup_{s\in[0,T]} D_s$ is the largest drawdown experienced over a specified period $[0, T]$. It is commonly used as a measure of the risk of holding the underlying asset over a period $[0, T]$.
- Maximum Drawdown MD_T is the worst possible loss that can arise over the period $[0, T]$ from first buying an asset and then selling it.



Maximum Drawdown and Drawup, and Their Derivatives (Cont'd)

- Similarly, maximum drawup $MU_T = \sup_{s\in[0,T]} U_s$ is the largest drawup experienced over $[0, T]$. It can be used as a measure of the return of holding the underlying asset over a period $[0, T]$.
- Maximum drawup MU_T is the best possible gain that can arise over the period $[0, T]$ from first buying an asset and then selling it.
- The success of volatility derivatives such as variance swaps suggests that derivatives on other risk measures such as maximum drawdown might be of interest.
- It is easy to suggest payoffs that depend on maximum drawdown or drawup. The problem in this day and age is to propose payoffs that can be easily understood and which can be robustly hedged.

Maximum Drawdown Insurance

- To introduce our suggested payoffs mathematically, we need to introduce two stopping times which are related to the Maximum Drawdown and the Maximum Drawup over $[0, T]$: for $K > 0$

$$T_D(K) = \inf\{t \geq 0 | D_t \geq K\} \qquad T_U(K) = \inf\{t \geq 0 | U_t \geq K\},$$
$$\{MD_T \geq K\} = \{T_D(K) \leq T\} \qquad \{MU_T \geq K\} = \{T_U(K) \leq T\}.$$

- We introduce two digital calls:
  1. Digital Call on Maximum Drawdown: an insurance against risk,
  $$DC_T^{MD}(K, T) = 1\{MD_T \geq K\} = 1\{T_D(K) \leq T\} \text{ for } K > 0.$$
  2. Digital Call on a Drawdown of at least K preceding a Drawup of the same size:
  $$DC_T^{D<U}(K, T) = 1\{T_D(K) \leq T_U(K) \wedge T\} \text{ for } K > 0.$$
- The first Digital Call clearly insures against a large Maximum Drawdown. The second Digital Call adds a contigency to this insurance which cheapens the premium.
- The two payoffs are easy to understand. The payoffs also act as building blocks for other derivatives written on Maximum Drawdown and Drawup.

What We Do

- Recall that we introduced two digital calls:
  1. Digital Call on Maximum Drawdown:
  $$DC_T^{MD}(K,T) = 1\{MD_T \geq K\} = 1\{T_D(K) \leq T\} \text{ for } K > 0.$$
  2. Digital Call on a Drawdown of at least K preceding a Drawup of the same size:
  $$DC_T^{D<U}(K,T) = 1\{T_D(K) \leq T_U(K) \wedge T\} \text{ for } K > 0.$$
- We first present a semi-static and semi-robust hedge of the Digital Call on Maximum Drawdown. The hedge uses Double-One-Touches.
- We then present a static robust hedge of Digital Call on the K-drawdown preceding a K-drawup. The hedge uses One-touch Knockouts.
- Finally, we present an alternative semi-static and semi-robust hedge of the Digital Call on Maximum Relative Drawdown. The hedge uses One-touch Knockouts and One-touches.

Hedging Instruments for Digital Call on Maximum Drawdown

- A Double-One-Touch (DOT) is a double barrier digital option with a higher barrier $H > F_0$ and a lower barrier $L < F_0$.
- A DOT pays $1 at expiry T if the underlying hits either barrier H or L before time T.
- Assuming no arbitrage, the price of a DOT at time t before its expiry T is:

$$DOT_t(L, H, T) = B_t(T)\mathbb{Q}_t^T\{\tau_L \wedge \tau_H \leq T\},$$

  where $B_t(T)$ is the price of a zero coupon bond maturing at T and $\mathbb{Q}^T$ is the EMM generated by the bond.
- Double No Touches (DNT) trade liquidly in FX market and from in-out parity:

$$DOT_t(L, H, T) = B_t(T) - DNT_t(L, H, T).$$

Assumptions

- Recall that $T_D(K)$ is the first time that the running drawdown reaches $K > 0$. The target digital call pays \$1 at T if and only if $T_D(K) \leq T$.
- We assume that the running maximum is continuous over the time interval $t \in [0, T_D(K) \wedge T]$.
- When the maximum does move up, we assume that the risk-neutral probability that F will exit from $M_t - \triangle$ is the same as the risk-neutral probability that F will exit from $M_t + \triangle$, for any $\triangle \geq 0$.
- An example of dynamics with this property is the independent time-changed Bachelier Model: $dF_t = a_t dW_t$ with $F_0 > 0$ and $da_t dW_t = 0$. Notice that the normal volatility process a is completely unspecified.
- We refer to this assumption as MCAES (Maximum Continuous Arithmetic Exit Symmetry).

Replicating DC on MD using DOT's

Theorem

Under frictionless market and MCAES, no aribtrage implies that a Digital Call on Maximum Drawdown can be replicated with bonds and Double-One-Touches (DOT):

$$\begin{aligned} DC_t^{MD}(K,T) &= B_t(T)\mathbb{Q}_t^T\{T_D(K) \le T\} \\ &= 1(T_D(K) \le t)B_t(T) + 1(T_D(K) > t)DOT_t(M_t - K, M_t + K, T). \end{aligned}$$

for any $t \in [0,T]$ *and* $K > 0$.

- In words, a Digital Call on the Maximum Drawdown is replicated by always holding a Double One Touch centered at the running maximum M_t, and whose width is the strike K of the Digital Call.

Replicating DC on MD using DOT's

- Recall that a DC on MD can be replicated with DOT's:

$$\begin{aligned} DC_t^{MD}(K,T) &= B_t(T)\mathbb{Q}_t^T\{T_D(K) \le T\} \\ &= 1(T_D(K) \le t)B_t(T) + 1(T_D(K) > t)DOT_t(M_t - K, M_t + K, T). \end{aligned}$$

- Since $M_0 = F_0$, the hedge initially holds $DOT_0(F_0 - K, F_0 + K, T)$.
- If the maximum never rises above F_0 over $[0, T]$, then the $DOT_0(F_0 - K, F_0 + K, T)$ pays \$1 at T if the running drawdown reaches K before T and it pays zero otherwise.
- If the maximum rises above F_0, then the investor must roll up both barriers of the DOT so that they remain equidistant from the running maximum .
- The strategy is self-financing because the cash outflow required to bring the lower barrier nearer by dM when the running maximum increases infinitesimally is financed by the cash inflow received from pushing the upper barrier away by dM (given that AES is in fact holding at such times).
- After rolling up, the lower barrier of the DOT is exactly \$$K$ below the maximum to date, as at initiation.

Link to Lévy's Theorem

- The Bachelier model $dF_t = \sigma dW_t$ is a very special case of the dynamics we are considering.
- In the Bachelier model, we have Lévy's theorem

$$\sup_{s\in[0,t]} W_s - W_t \overset{law}{=} |W_t|$$

  where W is a standard Brownian motion starting at 0.
- It follows that

$$\sup_{t\in[0,T]} \left(\sup_{s\in[0,t]} W_s - W_t \right) \overset{law}{=} \sup_{t\in[0,T]} |W_t|,$$

- Hence,

$$\mathbb{Q}_0^T \left\{ \sup_{t\in[0,T]} \left(\sup_{s\in[0,t]} W_s - W_t \right) \geq K \right\} = \mathbb{Q}_0^T \left\{ \sup_{t\in[0,T]} |W_t| \geq K \right\},$$

- As $W_t = \frac{F_t - F_0}{a}$, this results says that a DC on MD has the same price as a DOT.

Perfect Substitutes for DOT's

- In some markets, DOT's do not trade or else they only trade with heavy frictions.
- If Double No Touches (DNT's) trade, then one can instead use them since without a model, a DOT can be replicated with a bond and a DNT:

$$DOT_t(L, H, T) = B_t(T) - DNT_t(L, H, T).$$

- If DNT do not trade, then one can instead use One-touch Knockouts (OTKO).
- Let $OTKO_t(L, H, T)$ be the value at time $t \in [0, T]$ of an One-touch Knockout, i.e. a claim pays \$$1\{\tau_L \leq \tau_H \wedge T\}$.
- Without a model, a DOT can be replicated with two OTKO's:

$$DOT_t(L, H, T) = OTKO_t(L, H, T) + OTKO_t(H, L, T).$$

Imposing Dynamical Structure to Gain Liquidity

- For many underlyings, double barrier options such as DOT's, DNT's, and OTKO's do not trade, or else they only trade with heavy frictions.
- In such markets, we can instead use single barrier one-touches, henceforth OT's, if we are willing to impose slightly more dynamical structure (i.e. take more model risk).
- We assume that while $t \leq T_D(K) \wedge T$, the running range (maximum less minimum) is continuous. When the range increases, the event $\{m_T < F_t - \triangle\}$ has the same risk-neutral probability as the event $\{M_T > F_t + \triangle\}$, for any $\triangle > 0$.
- An example of dynamics with this property is the independently time-changed Bachelier Model: $dF_t = a_t dW_t$ with $F_0 > 0$ and $da_t dW_t = 0$.
- We refer to this assumption as RCAHS (Range Continuous Arithmetic Hitting Symmetry).

Replicating One-touch Knockouts under RCAHS

- Recall that without a model, a DOT can also be replicated with two OTKO's:

$$DOT_t(L, H, T) = OTKO_t(L, H, T) + OTKO_t(H, L, T).$$

Proposition

Under frictionless market and the above RCAHS assumption, no arbitrage implies that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

Replication of One-touch Knockouts under RCAHS (Cont'd)

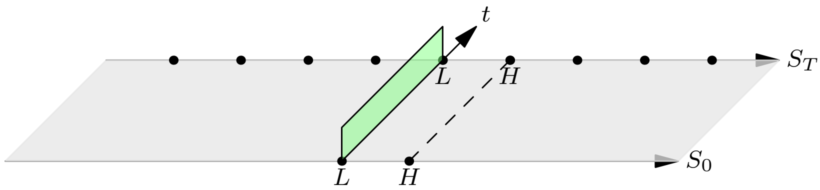
We want to prove that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits L first,



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Replication of One-touch Knockouts under RCAHS (Cont'd)

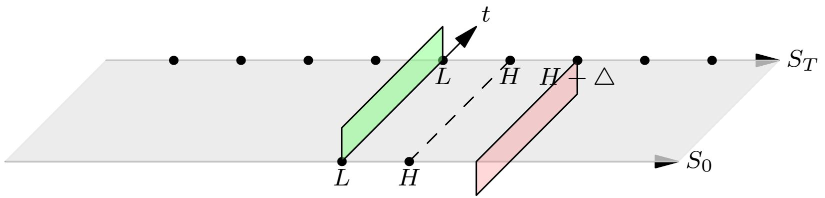
We want to prove that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits H first,



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Replication of One-touch Knockouts under RCAHS (Cont'd)

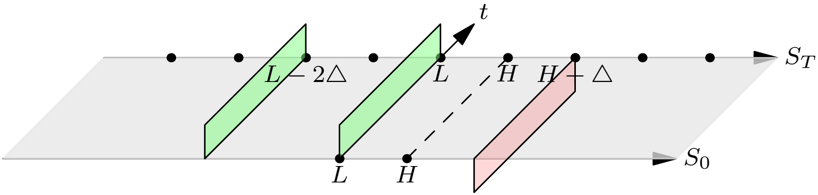
We want to prove that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits L first,



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Replication of One-touch Knockouts under RCAHS (Cont'd)

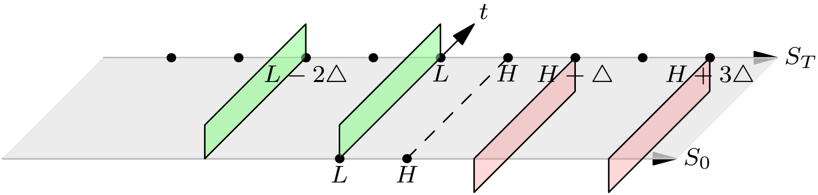
We want to prove that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits H first,



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Replication of One-touch Knockouts under RCAHS (Cont'd)

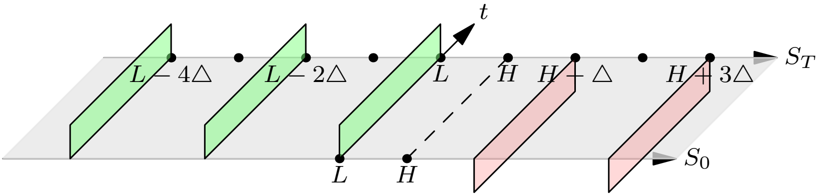
We want to prove that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L, H, T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H - (2n+1)\triangle, T) - OT_t(H + (2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits L first,



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Replicating DC on MD using OT's

- Recall that under frictionless market and MCAES, no aribtrage implies that:

$$\begin{aligned} DC_t^{MD}(K,T) &= B_t(T)\mathbb{Q}_t^T\{T_D(K)\le T\} \\ &= 1(T_D(K)\le t)B_t(T)+1(T_D(K)>t)DOT_t(M_t-K,M_t+K,T). \end{aligned}$$

  for any $t\in[0,T]$ and $K>0$.
- Without a model, a DOT can be replicated with two OTKO's:

$$DOT_t(L,H,T)=OTKO_t(L,H,T)+OTKO_t(H,L,T).$$

Theorem

Under frictionless market and RCAES, no aribtrage implies that a Digital Call on Maximum Drawdown can be replicated with bonds and One-Touches:

$$\begin{aligned} DC_t^{MD}(K,T) &= 1(T_D(K)\le t)B_t(T)+1(T_D(K)>t)\times \\ &\quad \left\{\sum_{n=0}^{\infty}[OT_t(M_t-(4n+1)K,T)+OT_t(M_t+(4n+1)K,T)]\right. \\ &\quad \left.-\sum_{n=1}^{\infty}[OT_t(M_t+(4n-1)K,T)+OT_t(M_t-(4n-1)K,T)]\right\}, \end{aligned}$$

for any $t\in[0,T]$ and $K>0$.

Contingent Protection against Maximum Drawdown

- So far, we have only examined derivatives whose payoff depends on Maximum Drawdown.
- An investor who understands the relevance of Maximum Drawdown as a risk measure would also understand the relevance of Maximum Drawup as a reward measure.
- A Digital Call on Maximum Drawdown struck at K only pays off if there is a drawdown of at least \$$K$. By adding the contingency that the drawdown occur before a drawup of the same size, the cost of this insurance is cheapened.
- Furthermore, we will show that the model risk that arises in hedging this contingent insurance can be eliminated.
- The reason is that the new payoff can be robustly replicated using One Touch Knockouts, which trade in FX markets.

Decomposing the Payoff of DC on Drawing Down before Drawing Up

- Let F_t, M_t and m_t be underlying price, the historical high and the historical low at time $t \in [0, T]$ of the underlying, respectively.
- On any path in the event $\{T_D(K) \leq T_U(K) \wedge T\}$, at $t < T_D(K) \wedge T_U(K)$,
  - If the spot does not reach a new high by $T_D(K)$, $M_{T_D(K)} = M_t$.
  - Otherwise, $M_{T_D(K)} \in (M_t, m_t + K)$.
- Replicate payoff based on the historical high when there is a crash: $M_{T_D(K)}$

$$
\begin{aligned}
1(T_D(K) \leq T_U(K) \wedge T) =& 1(T_D(K) = \tau_{M_{T_D(K)}-K} \leq T, M_{T_D(K)} \in [M_t, m_t + K)) \\
=& 1(\tau_{M_t-K} \leq T, M_{\tau_{M_t-K}} = M_t) \\
&+ \int_{M_t^+}^{(m_t+K)^-} 1(\tau_{H-K} \leq T)\delta(M_{\tau_{H-K}} - H)dH.
\end{aligned}
$$

- The two terms on the RHS arise as the payoff from traded instruments.

One-touch Knockouts and Their Spreads

- Consider an one-touch knockout with a (low) in-barrier L and a (high) out-barrier H. By no arbitrage, the value of this claim at any time t before expiry T is

$$OTKO_t(L,H,T) = B_t(T)\mathbb{Q}_t^T\{\tau_L \leq \tau_H \wedge T\} = B_t(T)\mathbb{Q}_t^T\{\tau_L \leq T, M_{\tau_L} < H\}$$

- A Ricochet-Upper-First Down-and-In claim is a spread of one-touch knockouts. It has a low barrier L and a high barrier H.

$$RUFDI_t(L,H,T) = \lim_{\epsilon \to 0^+} \frac{OTKO_t(L,H+\epsilon,T) - OTKO_t(L,H,T)}{\epsilon}$$
$$= B_t(T)E_t^{\mathbb{Q}^T}[1(\tau_L \leq T)\delta(M_{\tau_L} - H)].$$

It pays \$1 at expiry if and only if the spot grazes the upper barrier H and then hits L from above before T.

Robust Hedge of Digital Call on K-Drawdown preceding a K-Drawup

Theorem

Under frictionless markets, no arbitrage implies that the digital call on the K-drawdown preceding a K-drawup can be valued relative to the prices of bonds, one-touch knockouts and touch-upper-first down-and-in claims:

$$\begin{aligned} DC_t^{D<U}(K,T) =& 1(T_D(K) \leq T_U(K) \wedge t) B_t(T) + 1(t < T_D(K) \wedge T_U(K) \wedge T) \times \\ & \left\{ OTKO_t(M_t - K, M_t^+, T) + \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H-K, H, T) dH, \right\}. \end{aligned}$$

for any $t \in [0,T]$ and $K > 0$.

- The superscript $+$ on M in the 1st term on the 2nd line arises because:

$$OTKO_t(L, H^+, T) = B_t(T) \mathbb{Q}_t^T \{\tau_L \leq T, M_{\tau_L} \leq H\},$$

  while:

$$OTKO_t(L, H, T) = B_t(T) \mathbb{Q}_t^T \{\tau_L \leq T, M_{\tau_L} < H\}.$$

- Importantly, we make no assumption whatsoever concerning dynamics.
- One-touch knockouts do trade in FX markets.

Robust Hedge of Digital Call on K-Drawdown preceding a K-Drawup (Cont'd)

Recall that

$$DC_t^{D<U}(K,T) = 1(T_D(K) \leq T_U(K) \wedge t)B_t(T) + 1(t < T_D(K) \wedge T_U(K) \wedge T) \times \left\{ OTKO_t(M_t - K, M_t^+, T) + \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H-K, H, T)dH, \right\}.$$

for any $t \in [0, T]$ and $K > 0$.

- If the maximum never increase for any $r \in [t, T_D(K) \wedge T_U(K) \wedge T]$, then both sides pay out one dollar at T if the spot S drops below $M_t - K$.
- If the maximum does increase during the period $[t, T_D(K) \wedge T_U(K) \wedge T]$ and $T_U(K) \leq T_D(K) \wedge T$, then $M_{T_D(K)} \geq M_{T_U(K)} \geq m_t + K$. Both sides knock out.
- If the maximum increases over the period $[t, T_D(K) \wedge T_U(K) \wedge T]$ and $T_U(K) > T_D(K) \vee T$. Let t_L be the last instant in $[t, T_D(K) \wedge T_U(K) \wedge T]$ for which there was an increase. Then we reduce the problem after time t_L to the first case.
- In all cases, the payoff of the LHS is matched by the payoff of the RHS. No arbitrage implies both sides have the same value at all times.

Conclusions

- We introduced Digital Calls on Maximum Drawdown and on a Drawdown of at least K preceding a Drawup of the same size.
- These path-dependent payoffs can be used by investors to insure against drops in market prices.
- Digital Calls on Maximum Drawdown can be used to create any function of Maximum Drawdown, eg. an option.
- Under a weak assumption (MCAHS), the payoff to a Digital Call on Maximum Drawdown can be replicated by rolling up either Double One Touches, or equivalently two One Touch Knockouts (OTKO's)
- By adding the contingency that the drawdown occur before a drawup of the same size, the (slight) model risk can be eliminated since the contingent payoff can be robustly replicated using OTKO's and their spreads.
- By strengthening the continuity and symmetry assumptions, either digital call can instead be replicated with one-touches or vanillas.

Future Research

- We have a (long) paper containing all the details.
- To allow the range to jump, the paper models the underlying asset price as the difference of two doubly stochastic Poisson processes.
- To avoid the theoretical possibility of negative prices for the underlying asset, the paper also considers geometric models. In particular, (semi-) robust hedges are found when the underlying asset price is a geometric Brownian motion running on an independent unknown clock.
- To accommodate both nonnegative prices and strong negative skew, one can explore recent results (Carr and Nadtochiy 2009) on the construction of static hedges of barrier options on CEV processes.
- Since the real spot price process will never exactly satisfy any symmetry condition, one may always use the proposed "hedge" in this paper first, and then use a more realistic model to develop classical hedging strategies for the residual.