Inflation-Linked Pricing in the Presence of a Central Bank Reaction Function

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Introduction

We propose a pricing model for inflation-linked derivatives based on the premise that, to be successful, an inflation model has to take into account the central bank reaction function to explain the co-movement of interest rates and inflation.

To achieve this, we adapt elements of a mainstream macroeconomic model (the DSGE model with a Taylor rule) and price derivatives in a no-arbitrage setting. We formally prove that the no-arbitrage conditions hold in the inflation market and verify that the chosen macroeconomic model dynamics are consistent with the no-arbitrage framework.

The proposed approach is more ambitious than those currently most used in the industry (i.e. the Jarrow-Yildirim and the BGM-I modelling strategies) since the co-movement of interest rates and inflation is not taken as a given in our approach but is the result of central bank policy.

We propose a parsimonious strategy to calibrate the model to nominal interest rates, inflation term structures and smiles. We calibrate the model to recent market data and show that the calibration scheme is satisfactory.

The inflation market

Price indices quantify the evolution of price levels in the economy. Such indices are defined by considering the prices of representative baskets of goods and services, normalised to 100 at a given past date (the base).

We let denote I_t the level of the chosen price index at time t.

The realised inflation rate π_t for the interval $[s, t]$, expressed in an annual basis, is given by the annualised percentage growth rate on the price index:
$\pi_i = (\frac{I_t}{I_s} - 1)\frac{1}{t-s}$.

Inflation-linked instruments are traded assets for which the payoff depend on one (or more) price indices:

Plain Vanilla: Inflation-linked Bonds, Zero-coupon Inflation Swaps.

Options: Inflation Caps/Floors.

Exotics: Inflation Range Accruals, Inflation Spread Options, Inflation Hybrids.

The pricing problem

We wish to price inflation-linked contingent claims in a no-arbitrage setting.

We need (at least approximate) closed form expressions for nominal and inflation-linked bonds: this allows one to calibrate the model to the term structures of nominal rates and inflation.

Ideally we want to be able to reproduce the skew/smiles observed in vanilla option markets, both for nominal and inflation caps/floors.

We would like to keep the modelling framework as general as possible, and specify the distributions of the state variables only when necessary.

We want to start from a realistic description of the economy.

Background

The **Forex Analogy** is the cornerstone of most inflation-linked pricing models: it assumes the existence of both a nominal and a real interest rates system, each with a term structure. The price index plays the role of the exchange rate between the two systems: therefore all the FX pricing theory with stochastic rates can be easily "recycled". See Hughston (1998).

Jarrow-Yildirim (2003) propose to use two correlated Hull-White processes for the nominal and real short interest rates. The price index is modelled as a geometric brownian motion. However, real-rate volatilities or nominal/real correlations are difficult to mark in this model.

BGM-I. The BGM technology has been used to model the nominal and real term structures with more than one sources of randomness (as opposed to JY). Proposed in Mercurio (2005).

Hughston-Macrina (2008) assume the existence of a nominal and real pricing kernel and used a microeconomic approach based on the convenience yield of the money supply to determine the dynamics of the price index.

Market set-up

The problem is perhaps best approached in a discrete-time setting, where it is easier to control technicalities, and keep track of the underlying assumptions.

Let $\{t_i\}_{i=0,1,2,\ldots}$ denote a sequence of discrete times, not necessarily equally spaced, where t_0 is the present and $t_{i+1} > t_i$ for all $i \in \mathbb{N}_0$.

We assume the sequence $\{t_i\}$ is unbounded: for any given time T there exists a value of i such that $t_i > T$.

The market will be represented with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a "market filtration" $\{\mathcal{F}_i\}_{i \geq 0}$.

For simplicity, we often write $X_i = X_{t_i}$. This is not meant to suggest that the dates are equally spaced.

We assume that arbitrage is not possible in the market: if no probability measure is specified, the expectation is understood to be taken with respect to the real-world measure ($\mathbb{P}$). To perform a measure change from the physical measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$, we introduce the Radon-Nikodym derivative $\mu_i = (d\mathbb{Q}/d\mathbb{P})\ |_{t_i}$.

Pricing kernel - the properties

We assume that the economy is structured in a way such that arbitrage is not possible. We take the view that arbitrage-freeness is equivalent to the existence of a pricing kernel $\{\psi_i\}$. The pricing kernel has the following properties:

1. The process $\{\psi_i\}$ is a strictly positive supermartingale, with $\psi_0 = 1$.
2. The pricing kernel is given by $\psi_i = \prod_{j=1}^{i} (1 + \tau_j n_j)^{-1} \mu_i$, where μ_i is the Radon-Nikodym derivative $(d\mathbb{Q}/d\mathbb{P})\,|_{t_i}, \tau_i$ the year fraction and n_i the short-term nominal interest rate. Equivalently we can write: $\psi_i = \mu_i / B_i$.
3. For each non-dividend paying asset h paying the single cashflow H_N at time T_N, we have: $h_i \psi_i = \mathbb{E}^{\mathbb{P}}[\psi_N H_N | \mathcal{F}_i]$, i.e. $h_i \psi_i$ is a $\mathbb{P}$-martingale. If dividends $\{D_i\}$ are paid, this becomes $h_i \psi_i + \sum_0^i \psi_j D_j = \mathbb{E}^{\mathbb{P}}[\psi_N H_N + \sum_0^N \psi_j D_j | \mathcal{F}_i]$.
4. The pricing kernel $\{\psi_i\}$ is the inverse of the numeraire chosen to rescale the asset processes to $\mathbb{P}$-martingales.
5. The pricing kernel is related to nominal bond prices via the following: $P(t_i, t_{i+k}) = \mathbb{E}_i \psi_{i+k} / \psi_i$, $\forall k \in \mathbb{N}$.

Traded assets

We assume the existence of the following:

1. The short-term nominal interest rate n_i – set by the central bank – is the interest agreed at time t_{i-1} and paid at time t_i by the bank account on the balance at time t_{i-1}. The process $\{n_i\}$ is a previsible process, i.e. the short term nominal interest rate n_i is $\mathcal{F}_{i-1}$-measurable.
2. The bank account $B_i = \prod_{j=1}^{i}(1+\tau_j n_j)$, with $B_0 = 1$. Here τ_i represents the year fraction between times t_{i-1} and t_i. Since the interest rate n_i is $\mathcal{F}_{i-1}$-measurable, the bank account process $\{B_i\}$ is a previsible process. At time t_{i-1} the cash-flow that will occur at time t_i is already known: this is why the bank account is often referred to as the riskless asset.
3. A system of discount bonds $P(t_i, t_N)$, that pay one unit of currency at time t_N and have the following properties:
   - $P(t_i, t_N) = \mathbb{E}^{\mathbb{Q}}\left[\prod_{j=i+1}^{N}(1+\tau_j n_j)^{-1}\right]$
   - $P(t_i, t_i) = 1, \forall i$
   - $P(t_i, t_N) > 0, \forall i < N$
   - $\lim_{N\to\infty} P(t_i, t_N) = 0$
   - $P(t_i, t_{i+1}) = B_i / B_{i+1}$.

4. The price index process $\{I_i\}$ that describes the evolution over time of the price level.
5. A system of zero-coupon inflation index swaps (ZCIISs), such that the floating leg pays $(I_{i+M}/I_i) - 1$ and the fixed leg pays $(1 + X_M)^{M\tau_i} - 1$. The strikes X_i are quoted at time t_i for all maturities $t_M > t_i$.
6. A system of index-linked zero coupon bonds $P^I(t_i, t_M)$, which pay at maturity t_M the cash equivalent of the price index I_M. These bonds are quoted at time t_i for all maturities $t_M > t_i$ and have the following properties:
   - $P^I(t_i, t_M) = \mathbb{E}_i\left[\psi_M I_M\right]/\psi_i$
   - $P^I(t_i, t_i) = 1, \forall i$
   - $P^I(t_i, t_M) > 0, \forall i < M$.

FRN representation for the real pricing kernel

By using the set-up introduced so far, we can relate the pricing kernel ψ_i, the real rate r_i and the price index I_i via the following formula:

$$I_i = \frac{1}{\psi_i} \mathbb{E}_i \sum_{j=i+1}^{\infty} \tau_j r_j \psi_j I_j. \tag{1}$$

This condition is equivalent to

$$\psi_i^R = \mathbb{E}_i \sum_{j=i+1}^{\infty} \tau_j r_j \psi_j^R. \tag{2}$$

This result is model-independent, since no assumptions on the dynamics of interest rates, bond prices or price indices have been made.

Extension to illiquid assets

We show how the result (1) can be extended to any derivative written on a generic underlying index I_i. In fact no assumptions on the index were made, therefore I_i is not necessarily a price index, but can be any index whose value is known by market agents without ambiguity.

This makes the analysis presented above usable across different asset classes, regardless of how storable or tradable the index is. Examples are:

Electricity is not storable and a significant derivative market exists. We follow the same argument seen in the previous section and assume the existence of a market where electricity-index bonds are traded. This bond pays the electricity-index value at maturity, and can be rescaled by the current index level to obtain the electric bond. This instrument is then used to establish again the relationship (1) in the electricity market, which allows to write the product of the current electricity index and the current pricing kernel as the expected value of the infinite sum of its future values multiplied by the pricing kernel and the corresponding rate.

Property derivatives are swaps related to a property index, either residential or commercial. This market has significantly increased in size during the last five years, allowing investors to take a view on property without physically owning it.

Weather derivatives are used by companies whose cash-flows are dependent on the weather, like utilities, hotels, construction companies or golf clubs. The size of this market has not become significant as liquidity is very scant.

Longevity derivatives are a fledging asset class, where a coupon linked on the average life of a population is exchanged at maturity. Life insurance companies are long survival risk, whereas pension funds are short: investment banks can act as intermediaries and make some margins.

Emissions derivatives allow polluting companies to trade the emission credits in the market. The liquidity of this market is improving.

A model for the economy

The **inflation rate** is defined b $\pi_i = ((I_i/I_{i-1}) - 1)/\tau_i$. This is the annualised percentage growth rate of the price index.

The **output gap** x_i is defined by the difference between the actual and the potential log-linearised growth rate of the economy : $x_i = \hat{y}_i - \hat{y}_i^f$.

To provide a complete definition of the output gap, we introduce the GDP Y_i which is the value of all final goods and services produced in the economy between times t_{i-1} and t_i. The GDP annualised growth rate is as:
$y_i = ((Y_i/Y_{i-1}) - 1)/\tau_i$.

The growth rate y_i is assumed to have a long term equilibrium level $\bar{y}$ such that $\mathbb{E}(y_i) \to \bar{y}$ as $i \to +\infty$. The variable $\hat{y}_i$ is defined as the percentage deviation between the GDP growth rate y_i and its long term equilibrium level $\bar{y}$:
$\hat{y}_i = ((y_i/\bar{y}) - 1)$.

The economy is subject to some "inefficiencies": we introduce the potential GDP Y_i^f, which is defined as the GDP produced if there is no inefficiency: these inefficiencies prevent the actual GDP Y_i from reaching the "full employment" GDP Y_i^f.

Therefore we similarly derive the variables y_i^f, $\bar{y}_i^f$, and $\hat{y}_i^f$, which complete the definition of the output gap x_i.

If we assume that the processes $\{Y_i\}$, $\{Y_i^f\}$, and $\{I_i\}$ are adapted, the processes $\{x_i\}$ and $\{\pi_i\}$ are adapted too.

Other **economic assumptions** are made, including price stickiness, absence of government, optimizing behaviour of both consumers and firms and labour-based production function.

The **dynamics** of the output gap and inflation are determined in the model. They are referred to as the neo-keynesian demand curve and the neo-keynesian Phillips curve respectively. Their equations are:

$$x_i = \mathbb{E}_i x_{i+1} - \frac{1}{\sigma}(\hat{n}_{i+1} - \mathbb{E}_i \pi_{i+1}) + u_i \tag{3}$$

$$\pi_i = \beta \mathbb{E}_i \pi_{i+1} + k x_i. \tag{4}$$

DSGE dynamics and reaction function

Reaction Function. We assume that the central bank sets the short term nominal interest rate using a so-called Taylor rule:

$$n_{i+1} = \bar{n}_{i+1}(1 + \hat{n}_{i+1}) = \bar{n}_{i+1}(1 + \delta_x x_i + \delta_\pi \pi_i + v_i). \tag{5}$$

Here $\bar{n}_{i+1}$ is the equilibrium nominal interest rate at time t_i: δ_x and δ_π are the weights of the output gap and inflation in the central bank reaction function. Randomness is introduced by adding the stochastic term v_i.

We note that this formulation of the Taylor rule guarantees that the short term nominal interest rate process $\{n_i\}$ is previsible: in fact, the interest rate n_{i+1}, is paid at time t_{i+1} and set at time t_i given the inflation rate π_i, the output gap x_i, and a random term v_i. All these three variables are $\mathcal{F}_i$-measurable.

We add a third source of randomness z_i to improve calibration. This is a white noise process.

Calibration strategy - overview

We summarise the steps of the proposed calibration strategy in the Gaussian case (i.e. if the distribution of the shock processes $\{u_i\}$, $\{v_i\}$, and $\{z_i\}$ is Gaussian with zero mean and variance $\mathrm{Var}(u_i)$, $\mathrm{Var}(v_i)$, and $\mathrm{Var}(z_i)$ respectively):

1. The structural parameters of the DSGE model (β, σ, η, k, δ_x, δ_π) should be stable over time and are obtained from economic research.
2. The expectations of the output gap and inflation are provided by economic research but are likely to change over time.
3. The equilibrium short term nominal interest rate $\bar{n}_i$ is provided by economic research and is subject to frequent changes.
4. The variances $\mathrm{Var}(u_i)$, $\mathrm{Var}(v_i)$ and $\mathrm{Var}(z_i)$ are obtained by fitting the variances of nominal rates and inflation implied by the option markets (caps/floors).
5. The market price of risk processes $\{\bar{\lambda}_i^u\}$ and $\{\bar{\lambda}_i^v\}$ are calculated by fitting the market prices of nominal and real bonds, using the approximations for the bond prices and the inflation forwards.

Calibration - smiles

The DSGE dynamics allow one to derive the variance of the nominal interest rate and inflation.

$$\mathrm{Var}(\pi_i) = (K_2)^2(\sigma^2 \mathrm{Var}(u_i) + \mathrm{Var}(v_i)) + \mathrm{Var}(z_i) \tag{6}$$

$$\begin{aligned}\mathrm{Var}(n_{i+1}) = (\bar{n}_{i+1})^2(\delta^T K)^2\sigma^2\mathrm{Var}(u_i) + (\bar{n}_{i+1})^2(1-\delta^T K)^2\mathrm{Var}(v_i) + \\ +(\bar{n}_{i+1})^2\delta_\pi^2\mathrm{Var}(z_i)\end{aligned} \tag{7}$$

$$\begin{aligned}\mathrm{Cov}(\pi_i, n_{i+1}) = \bar{n}_{i+1}K_2(\delta^T K)\sigma^2\mathrm{Var}(u_i) + (\bar{n}_{i+1}K_2(\delta^T K - 1))\mathrm{Var}(v_i) + \\ +\bar{n}_{i+1}\delta_\pi\mathrm{Var}(z_i)\end{aligned} \tag{8}$$

These can be used to calibrate the variances of the random noise to market implied values.

Nominal rates and inflation caps/floors can be used to estimate normal (i.e. absolute) volatilities, allowing one to calculate the variances of the three processes $\{u_i\}$, $\{v_i\}$ and $\{v_i\}$ accordingly.

Calibration - nominal term structure

We take the variances of $\{u_i\}$, $\{v_i\}$ and $\{v_i\}$ as input into some approximated formulae for the nominal bond prices.

For example, if we assume the three processes are Gaussian, the bond price is approximately given by

$$P(t_{i+1}, t_{i+2}) \cong e^{c_1 + c_2 \mathrm{Var}(u_{i+1}) + c_3 \mathrm{Var}(v_{i+1}) + c_4 \mathrm{Var}(z_{i+1})} \tag{9}$$

where:

$$c_1 = -\tau_{i+2}\bar{n}_{i+2}(1 + \delta^T A \mathbb{E}_{i+1}\xi_{i+2}(\lambda_{i+1}) + \delta^T A I \lambda_{i+1})$$

$$c_2 = -\frac{1}{2}\left(\frac{\lambda^u_{i+1}}{\phi^u_{i+1}}\right)^2 + \frac{1}{2}\left(\frac{\lambda^u_{i+1}}{\phi^u_{i+1}} - \tau_{i+2}\bar{n}_{i+2}\delta^T K \sigma\right)^2$$

$$c_3 = -\frac{1}{2}\left(\frac{\lambda^v_{i+1}}{\phi^v_{i+1}}\right)^2 + \frac{1}{2}\left(\frac{\lambda^v_{i+1}}{\phi^v_{i+1}} + \tau_{i+2}\bar{n}_{i+2}(\delta^T K - 1)\right)^2$$

$$c_4 = \frac{1}{2}(\tau_{i+2}\bar{n}_{i+2}\delta_\pi)^2$$

We take the nominal bond prices observed in the market in order to calibrate to the nominal term structure.

We take the expectations for output gap and inflation, and equilibrium interest rates as an input.

The outputs are the market prices of risk for the processes $\{u_i\}$ and $\{v_i\}$, $\left\{\bar{\lambda}_i^u\right\}$ and $\left\{\bar{\lambda}_i^v\right\}$ respectively.

Calibration - inflation term structure

Under normality assumption, a similar formula is found for the real bond.

$$P^R(t_{i+1}, t_{i+2}) \cong \mathbb{E}_i^{\mathbb{P}} \left(e^{\tau_{i+1}\pi_{i+1} - \tau_{i+2} n_{i+2} + \bar{\lambda}_{i+1}^v v_{i+1} - \nu_{i+1}^v(\bar{\lambda}_{i+1}^v) + \bar{\lambda}_{i+1}^u u_{i+1} - \nu_{i+1}^u(\bar{\lambda}_{i+1}^v)} \right) \tag{10}$$

We define the forward inflation index $I_{i,i+1}^* \equiv P^R(t_{i+1}, t_{i+2})/P(t_{i+1}, t_{i+2})$.

$$I_{i,i+1}^* \cong e^{b_1 + b_2 \mathrm{Var}(u_{i+1}) + b_3 \mathrm{Var}(v_{i+1}) + b_4 \mathrm{Var}(z_{i+1})} \tag{11}$$

where

$$b_1 = \tau_{i+1} A_{2,1} \mathbb{E}_{i+1} x_{i+2} + \tau_{i+1} A_{2,2} (\mathbb{E}_{i+1} \pi_{i+2}(\lambda_{i+1}) + \lambda_{i+1}^v)$$

$$b_2 = -\frac{1}{2}\left(\frac{\lambda_{i+1}^u}{\phi_{i+1}^u}\right)^2 + \frac{1}{2}\left(\frac{\lambda_{i+1}^u}{\phi_{i+1}^u} + \tau_{i+1} K_2 \sigma\right)^2$$

$$b_3 = -\frac{1}{2}\left(\frac{\lambda_{i+1}^v}{\phi_{i+1}^v}\right)^2 + \frac{1}{2}\left(\frac{\lambda_{i+1}^v}{\phi_{i+1}^v} - \tau_{i+1} K_2\right)^2$$

$$b_4 = \frac{1}{2}(\tau_{i+1})^2.$$

Again the idea is to get the market prices of risk given the real term structure and the expectations.

Implementation and results

We implement the model in order to test its calibration to the market data of 13 January 2010: we took a snapshot of the EUR rates, EUR cap/floors volatilites, the HICPxT inflation ZCIISs, and HICPxT caps/floors volatilities. The model was calibrated with a monthly time step for the next 20 years.

DSGE parameters – We use standard choices for the DSGE structural parameters: since the ECB major concern is currently to stimulate growth and inflation seems to be subdued, our choice was for a δ_x much higher than δ_π.

Parameter	**Description**	**Indicative Level**
σ	Consumption elasticity	1.05
k	Market flexibility	0.015
ϱ (monthly)	Subjective discount rate	0.03
ω (monthly)	Price stickiness	0.9175
δ_π	Reaction to inflation	1
δ_x	Reaction to output gap	2.5

Expectactions and equilibrium rate – We have used input inflation expectations and equilibrium rates derived from market implied data, in such a way that the difference between implied and the expected is never greater than 40 basis points (1 bp $= 1/10000$).

Volatilities – We assume that shocks are normally distributed: therefore the distribution of the short rate and inflation is still Gaussian. This is a distinct advantage, because the normal distribution for the rates level is very often used by market practitioners to model market skews.

At each time step, we calibrate the volatilities of u_i, v_i and z_i in order to match the market implied forward volatilities of the Libor and inflation rate.

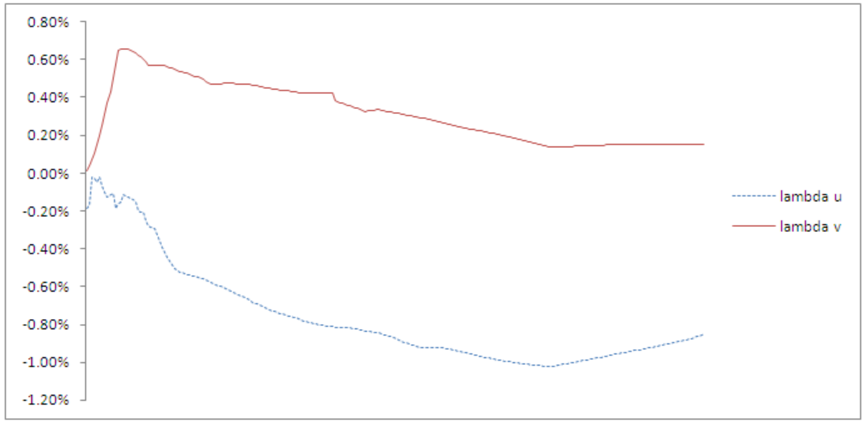
Once the market implied forward variances $\mathrm{Var}(n_i)$ and $\mathrm{Var}(\pi_i)$ have been calculated a numerical minimisation algorithm is used to find the implied $\mathrm{Var}(u_i)$, $\mathrm{Var}(v_i)$ and $\mathrm{Var}(z_i)$ by using the variance formulas (7) and (6).

There are two linear equations in three unknowns, therefore allowing for an infinite number of solutions: this is not a major concern as the numerical minimisation is straightforward and gives consistent curves for the variances

$\mathrm{Var}(u_i)$, $\mathrm{Var}(v_i)$ and $\mathrm{Var}(z_i)$.

Overall we think that obtaining linear closed forms for the variances is a useful achievement that makes this model tractable.

Market prices of risk – Once the variances $\mathrm{Var}(u_i)$, $\mathrm{Var}(v_i)$ and $\mathrm{Var}(z_i)$ are calibrated, we calibrate to the nominal term structure and to the inflation swap curve by searching for the market prices of risk λ_i^u and λ_i^v . Their shapes are smooth, showing that the numerical minimisation problem yields sensible results.



Final comments

The model shows high analytical tractability.

The model avoids the need for the exogenous specification of the real rate.

Approximate closed forms are available in the Gaussian case. This makes the calibration smooth.

The model aims at a realistic mathematical description of the economy.

The co-movement of nominal interest rates and inflation is explicitly derived from the central bank reaction function:

$$\mathrm{Cov}(\pi_i, n_{i+1}) = \bar{n}_{i+1} K_2(\delta^T K)\sigma^2 \mathrm{Var}(u_i) + (\bar{n}_{i+1} K_2(\delta^T K - 1))\mathrm{Var}(v_i) + \tag{12}$$
$$+\bar{n}_{i+1}\delta_\pi \mathrm{Var}(z_i).$$

Our view is that, in the future, the modelling of interest rates derivatives will benefit from using macroeconomic relationships to explain the dynamics of the price levels and central bank policy.