A Multiscale Model with Stochastic Elasticity

Jeong-Hoon Kim[1], Jungwoo Lee[1], Suk-Hyun Yu[2], Song-Ping Zhu[3]

[1]Department of Mathematics, Yonsei University
[2]KB Kookmin Bank, Seoul, Korea
[3]School of Mathematics and Statistics, University of Wollongong

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Motivation

•Although Black and Scholes assumed that the volatility is constant over all strike prices and maturities, in real market, the implied volatility curve(or surface) shows a curvature;
•Volatility level shows a negative correlation with the underlying asset price.

The CEV(Constant Elasticity of Variance) Model

CEV

First introduced by Cox(1975), Cox and Ross(1976).

The model assumes that the dynamics of the underlying is given as follows:

$$dX_t = \mu X_t dt + \sigma_0 X_t^{\theta/2} dW_t,$$

where θ is responsible for the elasticity of variance which is assumed to be constant.

Option Price under CEV Model

Under this model, it is known that the option price is given by the following formula(Schroder(1989)):

CEV Options Price

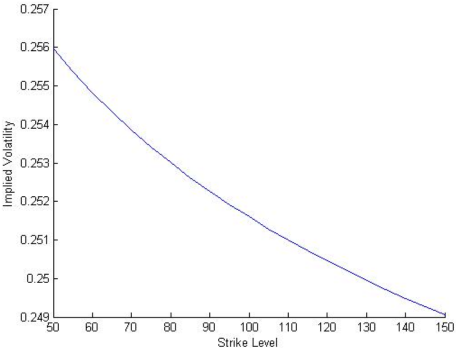
$$C_{CEV}(t, X_t) = X_t \sum_{n=0}^{\infty} g(n+1, x) G(n+1+\frac{1}{2-\theta}, kK^{2-\theta})$$
$$-Ke^{-r(T-t)} \sum_{n=0}^{\infty} g(n+1+\frac{1}{2-\theta}, x) G(n+1, kK^{2-\theta}),$$

where

$$g(m, v) = \frac{e^{-v} v^{m-1}}{\Gamma(m)}, \qquad G(m, v) = \int_v^{\infty} g(m, u) du,$$
$$k = \frac{2r}{\sigma_0^2 (2-\theta)(e^{r(2-\theta)(T-t)} - 1)}, \quad x = kX_t^{2-\theta} e^{r(2-\theta)(T-t)}$$

Benefits from CEV

It captures the volatility skew phenomenon.



Downsides of CEV

It tells us a false dynamics of implied volatility which may lead unstable hedges.
The correct dynamics of volatility is such that the volatility curve shifts in the same direction of the underlying asset movement(Hagan(2002)).

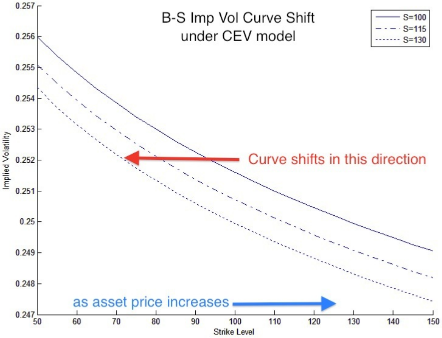


Figure: Wrong dynamics by CEV

The Stochastic Elasticity of Variance Model

The SEV(Stochastic Elasticity of Variance) Model

Introduction of the Stochastic Elasticity

The elasticity of variance parameter in CEV model : $\theta/2$

Some empirical studies show that $\theta > 2$ (Bollerslev et al.(1988), Campbell and Hentschel(1992), Ghysels et al.(2005)).

Other studies show that $\theta < 2$ (Campbell(1987), Breen et al(1989), Glosten et al(1993), Brandt and Kang(2004)).

It also is shown to be time-varying (Harvey(2001), Ghysels et al(1996)).

The SEV(Stochastic Elasticity of Variance) Model

CEV to SEV

$$\theta/2 \Longrightarrow 1 - \gamma f(Y_t)$$

SEV Model dynamics:

Our model is governed by the SDEs

$$dX_t = \mu X_t dt + \sigma X_t^{1-\gamma f(Y_t)} dB_t^x, \qquad (2.1)$$

$$dY_t = \alpha(m - Y_t)dt + \beta dB_t^y, \qquad (2.2)$$

with assumption that the Brownian motions B_t^x and B_t^y are correlated each other and f is a bounded function.

Notice that as γ approaches to zero, the model becomes that of Black and Scholes. Hence, we expect the Black-Scholes price to be the leading-order term in our solution.

Multiple Scales

Assumption

Y_t has an invariant distribution with variance $\frac{\beta^2}{2\alpha}$.
Assume that $\nu := \frac{\beta}{\sqrt{2\alpha}}$ is an $O(1)$-term.

Assumptions on Multiple Scales

•Fast mean-reverting of the process Y_t. (*i.e.* α being large.)
•$1-\gamma f(Y_t)$(elasticity) becomes stable to be 1. (*i.e.* γ being small.)
•α being large while γ being small.

Multiple Scales

Small parameters for the job

$$\epsilon = \frac{1}{\alpha},$$
$$\delta = \gamma^2$$

Motivation

δ: We want to have the Black-Scholes price as a leading order term.

ϵ: Fast mean-reverting OU process is analytcally more tractable.

Change of Measure

Our SDEs under the Risk-Neutral Measure

Under risk-neutral measure, they become

$$dX_t = rX_t dt + \sigma X_t^{1-\sqrt{\delta} f(Y_t)} dW_t^x, \tag{3.1}$$

$$dY_t = \left[\frac{1}{\epsilon}(m - Y_t) - \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\Lambda^{\delta}(X_t, Y_t)\right] dt + \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\, dW_t^y \tag{3.2}$$

where the correlation of Brownian motions W_t^x and W_t^y are given by

$$d\langle W^x, W^y\rangle_t = \rho dt$$

The Option Price

The Option Price

Under the risk-neutral measure, the option price is given by

$$P(t, X_t, Y_t) = E^*[e^{-r(T-t)}h(X_T)|X_t, Y_t], \tag{3.3}$$

where $h(X_T)$ is the corresponding payoff function for either put or call option.

Feynman-Kac's Formula

From the Feynman-Kac's formula, one can obtain the pricing PDE

$$\begin{aligned} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} &= 0 \\ \mathcal{L}^{\epsilon,\delta} &:= \frac{\partial}{\partial t} + \mathcal{L}^{\epsilon,\delta}_{X,Y} - r\cdot, \\ P^{\epsilon,\delta}(T,x,y) &= h(x), \end{aligned} \tag{3.4}$$

where $\mathcal{L}^{\epsilon,\delta}_{X,Y}$ is the infinitesimal generator of the diffusion process (X_t, Y_t) given by

$$\begin{aligned} \mathcal{L}^{\epsilon,\delta}_{X,Y} = &\frac{1}{2}\sigma^2 x^{2(1-f(y)\sqrt{\delta})}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} \\ &+ \frac{1}{\sqrt{\epsilon}}\left(\rho\nu\sqrt{2}\sigma x^{1-f(y)\sqrt{\delta}}\frac{\partial^2}{\partial x \partial y} - \nu\sqrt{2}\Lambda^{\delta}(x,y)\frac{\partial}{\partial y}\right) \\ &+ \frac{1}{\epsilon}\left(\nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}\right). \end{aligned} \tag{3.5}$$

Asymptotic Analysis

Singular perturbation for ϵ and regular perturbation for δ.

Separating the Operator $\mathcal{L}^{\epsilon,\delta}$

$$\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^{\delta} + \mathcal{L}_2^{\delta}, \tag{4.1}$$

$$\mathcal{L}_0 := \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{4.2}$$

$$\mathcal{L}_1^{\delta} := \rho\nu\sqrt{2}\sigma x^{1-f(y)\sqrt{\delta}} \frac{\partial^2}{\partial x \partial y} - \nu\sqrt{2}\Lambda^{\delta}(x,y)\frac{\partial}{\partial y}, \tag{4.3}$$

$$\mathcal{L}_2^{\delta} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^{2(1-f(y)\sqrt{\delta})}\frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot) \tag{4.4}$$

Regular Perturbation for δ

Handling Dependencies of $\mathcal{L}_1^\delta$ and $\mathcal{L}_2^\delta$ on δ

$$\mathcal{L}_1^\delta := \rho\nu\sqrt{2}\sigma x^{1-f(y)\sqrt{\delta}}\frac{\partial^2}{\partial x \partial y} - \nu\sqrt{2}\Lambda^\delta(x,y)\frac{\partial}{\partial y},$$

$$\mathcal{L}_2^\delta := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^{2(1-f(y)\sqrt{\delta})}\frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot)$$

Use the expansion

$$x^{1-f\sqrt{\delta}} = x(1 - \sqrt{\delta} f \log x + \delta\frac{(f\log x)^2}{2!} - \delta\sqrt{\delta}\frac{(f\log x)^3}{3!} + \cdots)$$

Asymptotic Analysis

Further Expansion in Powers of $\sqrt{\delta}$

$$\mathcal{L}_1^\delta = \mathcal{L}_{10} + \sqrt{\delta}\mathcal{L}_{11} + \delta\mathcal{L}_{12} + \cdots, \tag{4.5}$$

$$\mathcal{L}_{10} := \nu\sqrt{2}\rho\sigma x \frac{\partial^2}{\partial x \partial y}, \tag{4.6}$$

$$\mathcal{L}_{11} := -\nu\sqrt{2}\rho\sigma x f \log x \frac{\partial^2}{\partial x \partial y}, \tag{4.7}$$

$$\mathcal{L}_{12} := \nu\sqrt{2}\rho\sigma x \frac{(f \log x)^2}{2!} \frac{\partial^2}{\partial x \partial y}, \tag{4.8}$$

$$\vdots$$

Asymptotic Analysis

Further Expansion in Powers of $\sqrt{\delta}$ -cont

and

$$\mathcal{L}_2^\delta = \mathcal{L}_{20} + \sqrt{\delta}\mathcal{L}_{21} + \delta\mathcal{L}_{22} + \cdots, \tag{4.9}$$

$$\mathcal{L}_{20} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot) = \mathcal{L}_{BS}, \tag{4.10}$$

$$\mathcal{L}_{21} := -\sigma^2 x^2 f \log x \frac{\partial^2}{\partial x^2}, \tag{4.11}$$

$$\mathcal{L}_{22} := \sigma^2 x^2 (f \log x)^2 \frac{\partial^2}{\partial x^2}, \tag{4.12}$$

$$\vdots$$

Asymptotic Analysis

Asymptotic Analysis

We plug the expansion

$$P^{\epsilon,\delta} = P_0^\epsilon + \sqrt{\delta} P_1^\epsilon + \delta P_2^\epsilon + \cdots \tag{4.13}$$

into the PDE (3.4) and obtain the following hierarchy:

$$\delta^0 : \quad (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{10} + \mathcal{L}_{20})P_0^\epsilon = 0, \tag{4.14}$$

$$\delta^{\frac{1}{2}} : \quad (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{10} + \mathcal{L}_{20})P_1^\epsilon + (\frac{1}{\sqrt{\epsilon}}\mathcal{L}_{11} + \mathcal{L}_{21})P_0^\epsilon = 0, \tag{4.15}$$

$$\begin{aligned} \delta : \quad & (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{10} + \mathcal{L}_{20})P_2^\epsilon + (\frac{1}{\sqrt{\epsilon}}\mathcal{L}_{11} + \mathcal{L}_{21})P_1^\epsilon \\ & + (\frac{1}{\sqrt{\epsilon}}\mathcal{L}_{12} + \mathcal{L}_{22})P_0^\epsilon = 0, \end{aligned} \tag{4.16}$$

$$\vdots$$

and impose the final conditions $P_0^\epsilon(T,x,y) = h(x)$ and $P_1^\epsilon(T,x,y) = 0 \ldots$

Asymptotic Analysis

Analysis of δ^0 Power w.r.t. the Parameter ϵ

Expand P_0^ϵ as follows

$$P_0^\epsilon = P_{0,0} + \sqrt{\epsilon} P_{0,1} + \epsilon P_{0,2} + \epsilon\sqrt{\epsilon} P_{0,3} + \cdots$$

with the final conditions $P_{0,0}(T,x,y) = h(x)$, $P_{0,1}(T,x,y) = 0$, $P_{0,2}(T,x,y) = 0 \ldots$

Then we have the following sequence of PDEs:

$$\begin{aligned} \mathcal{L}_0 P_{0,k} + \mathcal{L}_{10} P_{0,k-1} + \mathcal{L}_{20} P_{0,k-2} &= 0, \\ P_{0,-2} &:= 0, \\ P_{0,-1} &:= 0, \end{aligned} \tag{4.17}$$

where $k = 0, 1, 2, \cdots$.

Asymptotic Analysis

A Few More Assumptions

For $k = 0$, the PDE,

$$\mathcal{L}_0 P_{0,0} = 0,$$

is assumed to admit only solutions that do not grow so much as

$$\frac{\partial P_{0,0}}{\partial y} \sim e^{\frac{y^2}{2}}, \quad y \to \infty,$$

then $P_{0,0}$ becomes a function of t and x only; $P_{0,0} = P_{0,0}(t, x)$

Similar assumption for $k = 1$ gives the same result on $P_{0,1}$; $P_{0,1} = P_{0,1}(t, x)$.

Asymptotic Analysis

A Few More Assumptions -cont

We impose the centering condition on $\mathcal{L}_{2,0}$, *i.e.*

$$\langle \mathcal{L}_{20} P_{0,0} \rangle = \mathcal{L}_{20} P_{0,0} = 0,$$

where

$$\langle g \rangle := \frac{1}{\sqrt{2\pi\nu^2}} \int_{\infty}^{\infty} g(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy$$

in order to admit the solutions only with reasonable growth.

Asymptotic Analysis

A Few More Assumptions -cont

We obtain the final value problem:

$$\mathcal{L}_{20}P_{0,0} := \mathcal{L}_{BS}P_{0,0} = 0,$$

$$P_{0,0}(T,x,y) = h(x) \tag{4.18}$$

thus

$$\mathcal{L}_0 P_{2,0} = 0 \tag{4.19}$$

As we repeat this methodology, and properly apply the centering condition, we obtain that $P_{0,0}$ is the Black-Scholes price and $P_{0,1} = 0$, and $P_{0,2} = 0$.

Asymptotic Analysis

Analysis of $\delta^{\frac{1}{2}}$ Power w.r.t. the Parameter ϵ

Expand P_1^ϵ as follows

$$P_1^\epsilon = P_{1,0} + \sqrt{\epsilon} P_{1,1} + \epsilon P_{1,2} + \epsilon\sqrt{\epsilon} P_{1,3} + \cdots$$

with the final conditions $P_{1,0}(T,x,y) = h(x)$, $P_{1,1}(T,x,y) = 0$, $P_{2,1}(T,x,y) = 0 \ldots$

Asymptotic Analysis

As we apply the same logic, we obtain the following equation:

$$\begin{aligned} &\mathcal{L}_{BS} P_{1,0}^{\delta} = V_{1,0}^{\delta} x^2 \log x \frac{\partial^2 P_{0,0}}{\partial x^2}, \\ &P_{1,0}^{\delta}(T, x, y) = 0, \\ &V_{1,0}^{\delta} := \gamma \sigma^2 \langle f \rangle. \end{aligned} \tag{4.20}$$

The Group Parameter: $V_{1,0}^{\delta}$

The leading order correction is entirely captured by the group parameter $V_{1,0}^{\delta} := \gamma\sigma^2\langle f\rangle$ that we introduced earlier.
There is no need for the knowledge of the parameters δ, ϵ, ν, m, ρ, and specific form of the function f.
$V_{1,0}^{\delta}$ can be obtained by calibrating the model to the option prices on the market.

Integral Representation

The price $P^{\epsilon,\delta}$ as a solution of the final value problem (4.20) with $h(x) = (x-K)^+$(*i.e.* for a call option) is asymptotically given by

$$\begin{aligned} P^{\epsilon,\delta} \sim\ & x\Phi(d_1) - K\Phi(d_2) \\ & -V_{1,0}^{\delta}(K/\sigma^2)\int_0^{\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi s(\tau-s)}}e^{-\frac{(\tilde{y}-\tilde{x})^2}{4(\tau-s)}+\tilde{x}} \\ & *(\tilde{y}+\log(K))\phi(d_1)e^*(\tilde{y}-\tilde{x},\tau-s)d\tilde{y}ds, \end{aligned} \tag{4.21}$$

where

$$\tilde{x} := \log(x/K), \qquad \tau := \frac{1}{2}\sigma^2(T-t),$$

$$e^*(x,t) := e^{\frac{1}{2}(k+1)x-\frac{1}{4}(k+1)^2t}, \quad k := \frac{2r}{\sigma^2},$$

and Φ is the cumulative normal distribution and ϕ is the standard normal density function and d_1, d_2 are as appeared in Black-Scholes model.

Integral Representation

Proof

One can solve PDE (4.20) for $P_{1,0}^{\delta}$ by transforming it into the heat equation with a source term via the transformation

$$\tilde{x} := \log(x/K), \quad \tau := \frac{1}{2}\sigma^2(T-t)$$

for independent variables and the transformation

$$u(\tau, \tilde{x}) = \frac{P_{1,0}^{\delta}}{K} e^{\frac{1}{2}(k-1)\tilde{x} + \frac{1}{4}(k+1)^2\tau}$$

for dependent variable.

Integral Representation

Proof-cont

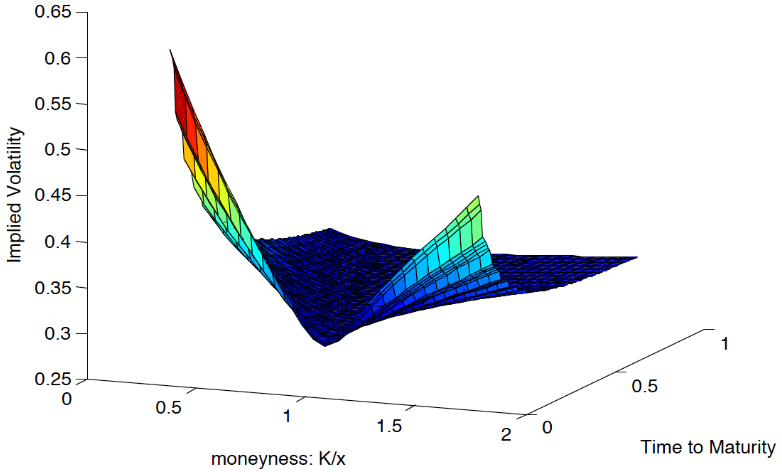
Since $\frac{\partial^2 P_{0,0}}{\partial x^2}$ is merely the Gamma of an option, the resultant equation for u as follows:

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial \tilde{x}^2} - V_{1,0}^{\delta} \frac{\sqrt{2}}{\sigma^2 \sqrt{\tau}} (\tilde{x} + \log K) \phi(d_1) e^{\frac{1}{2}(k+1)\tilde{x} + \frac{1}{4}(k+1)^2 \tau} \\ u(0, \tilde{x}) &= 0. \end{aligned}$$

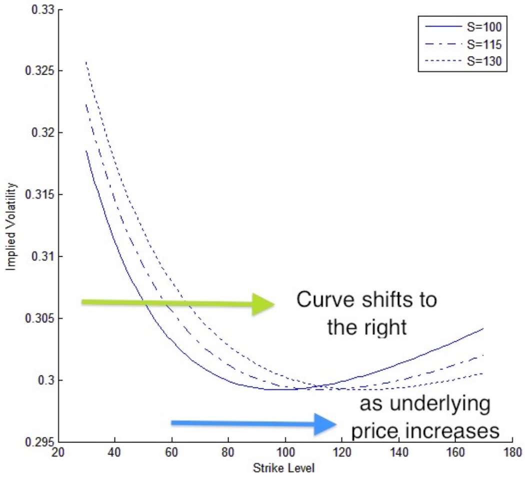
This is the heat equation with zero initial condition and a nonzero source term and its solution u and subsequently $P_{1,0}^{\delta}$ is well-known.

Some Simulation Results

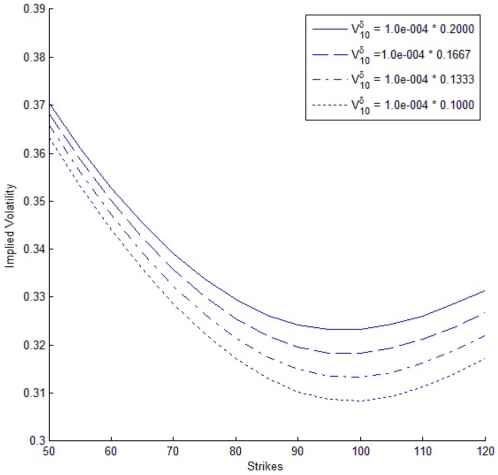
The Implied Volatility Surface



Dynamics of Implied Volatility



The Implied Volatility Curves with Various $V^{\delta}_{1,0}$'s



Conclusion

Conclusion and Works to be done

•SEV model overcomes the major drawback of the Black-Scholes model and gives us a smile curve.
•It fits observed market behavior of volatility curve shift and overcomes problems that CEV model has.
•A model with the CEV price in the leading order term.

Thank you for your attention!

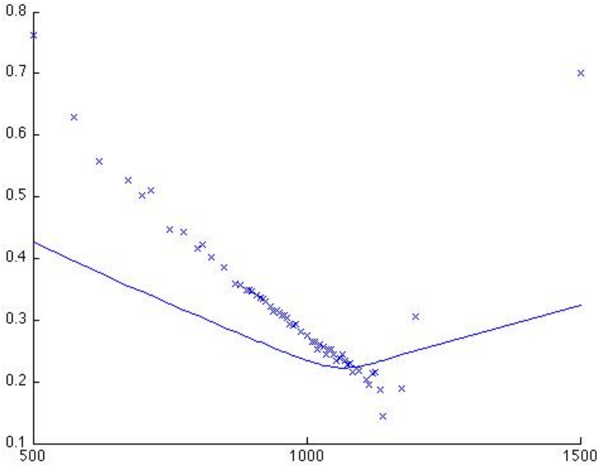


Figure: Fitted implied volatility curve with Spot: S&P500