Pricing and Hedging with Constant Elasticity and Stochastic Volatility

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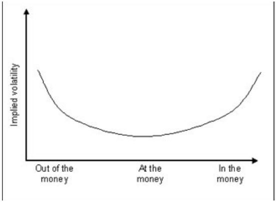
Outline

Drawback of Black Scholes Model

- Smile curve



- Rubinstein (1985)
- Jackwerth and Rubinstein (1996)
- In B.S. Model, Implied Volatility curve is flat.
- We need to use the implied volatility which explicitly depends on the option strike and maturity.

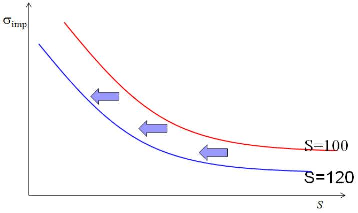
Local Volatility Model

Local Volatility Model

- One needs volatility to depend on underlying
- Local Volatility Models

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$$

- The dynamics of Implied Volatility in Local Volatility model.



This is opposite to real market.(Hagan, 2002)

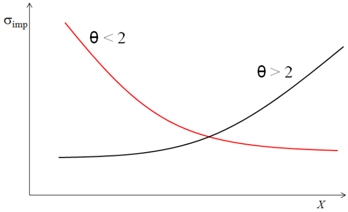
CEV Model

- CEV (constant elasticity of variance) diffusion model
- X_t stock price s.t.

$$dX_t = \mu X_t dt + \sigma X_t^{\frac{\theta}{2}} dW_t$$

- Introduced by Cox and Ross(1976)
- Studied by Beckers (1980), Schroder(1989), Boyle and Tian(1999), Davydov and Linetsky(2001), Delbaen and Schirakawa(2002), Heath and Platen(2002), Carr and Linetsky(2006) and etc.
- Analytic tractability

CEV Model cont.



- When $\theta = 2$ the model is Black-Scholes case.
- When $\theta < 2$ volatility falls as stock price rises.
  $\Rightarrow$ realistic, can generate a fatter left tail.
- When $\theta > 2$ volatility rise as stock price rises.
  $\Rightarrow$ (futures option)

CEV Model cont.

Theorem (Lipton ,2001)

The call option price C_{CEV} for $X_t = x$ is given by

$$\begin{aligned} C_{CEV}(t,x) &= e^{-r(T-t)} x \int_{\check{K}}^{\infty} \left(\frac{\check{x}}{y}\right)^{\frac{1}{2(2-\theta)}} e^{-(\check{x}+y)} I_{\frac{1}{2-\theta}}(2\sqrt{\check{x}y})\, dy \\ &+ e^{-r(T-t)} K \int_{\check{K}}^{\infty} \left(\frac{y}{\check{x}}\right)^{\frac{1}{2(2-\theta)}} e^{-(\check{x}+y)} I_{\frac{1}{2-\theta}}(2\sqrt{\check{x}y})\, dy, \end{aligned}$$

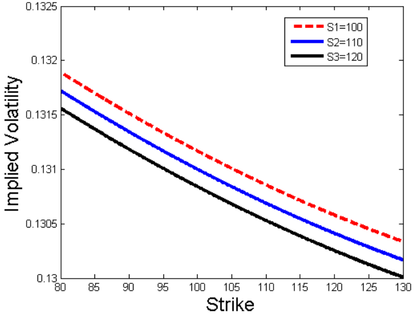
where

$$\check{x} = \frac{2xe^{r(2-\theta)(T-t)}}{(2-\theta)^2\chi}, \qquad \chi = \frac{\sigma^2}{(2-\theta)r}\left(e^{r(2-\theta)T} - e^{r(2-\theta)t}\right)$$

$$\check{K} = \frac{2K^{2-\theta}}{(2-\theta)^2\chi}$$

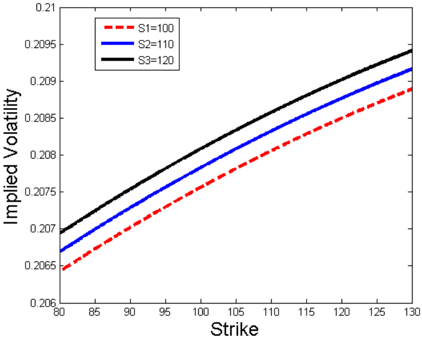
Dynamics of Implied Volatility for CEV model

$\theta = 1.9$



Dynamics of Implied Volatility for CEV model cont.

$\theta = 2.1$



Outline

Stochastic Volatility CEV model

Stochastic Volatility CEV model

Underlying asset price Model X_t and Y_t

$$dX_t = \mu X_t dt + f(Y_t) X_t^{\frac{\theta}{2}} dW_t \tag{1}$$
$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t, \tag{2}$$

where $f(y)$ smooth function and the Brownian motion $\hat{Z}_t$ is correlated with W_t such that

$$d < W, \hat{Z} >_t = \rho dt. \tag{3}$$

In terms of the instantaneous correlation coefficient ρ, we write

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t \tag{4}$$

Characteristics

- The new volatility is given by the multiplication of a function of a new process
- The new process is taken to be an Ito process (O-U process):

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t.$$

- α = rate of mean reversion.
  Assume that mean reversion is fast.
  So, α is large enough.

Corrected Pricing

Use Risk Neutral Valuation method

Equivalent martingale measure Q, Option price is given by the formula

$$P(t,x,y) = E^{Q}[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y] \tag{5}$$

Corrected Pricing Theorem

Theorem 3.1

The option price $P(t,x,y)$ defined by (5) satisfies the Kolmogorov PDE

$$P_t + \frac{1}{2}f^2(y)x^\theta P_{xx} + \rho\beta f(y)x^{\frac{\theta}{2}}P_{xy} + \frac{1}{2}\beta^2 P_{yy} + rxP_x + (\alpha(m-y) - \beta\Lambda(t,x,y))P_y - rP = 0, \quad (6)$$

where

$$\Lambda(t,x,y) = \rho\frac{\mu - r}{f(y)x^{\frac{\theta-2}{2}}} + \sqrt{1-\rho^2}\,\gamma(y).$$

Asymptotic theory

- Develop an asymptotic theory on fast mean reversion
- Introduce A small parameter ϵ

$$\epsilon = \frac{1}{\alpha}$$

- Assume $\nu = \dfrac{\beta}{\sqrt{2\alpha}}$ is fixed in scale as ϵ become zero.

$$\alpha \sim \mathcal{O}(\epsilon^{-1}), \;\; \beta \sim \mathcal{O}(\epsilon^{-1/2}), \;\; \text{and} \;\; \nu \sim \mathcal{O}(1).$$

Singular Perturbation

To solve the PDE (6), we use Singular Perturbation method.

Procedure

- Substituting the asymptotic series

$$P(x;\epsilon) \approx \sum_{n=0}^{\infty} \epsilon^n P_n(x)$$

  into the differential equation.
- Expanding all quantities in a power series in ϵ.
- Collecting terms with same powers of ϵ and equating them to zero.
- Solving this hierarchy of the problem sequentially.

Asymptotic theory

After rewritten in terms of ϵ, the PDE (6) becomes

$$P_t + \frac{1}{2} f^2(y) x^\theta P_{xx} + \rho \frac{\sqrt{2}\nu}{\sqrt{\epsilon}} f(y) x^{\frac{\theta}{2}} P_{xy} + \frac{1}{2}\frac{2\nu^2}{\epsilon} P_{yy}$$

$$+ rxP_x + (\frac{1}{\epsilon}(m-y) - \beta \frac{\sqrt{2}\nu}{\sqrt{\epsilon}} \Lambda(t,x,y)) P_y - rP = 0.$$

Collecting by ϵ order,

$$\frac{1}{\epsilon}\left(\nu^2 P_{yy} + (m-y)P_y\right) + \frac{1}{\sqrt{\epsilon}}(\sqrt{2}\rho\nu f(y) x^{\frac{\theta}{2}} P_{xy} - \sqrt{2}\nu\Lambda P_y)$$

$$+ (P_t + \frac{1}{2} f^2(y) x^\theta P_{xx} + rxP_x - rP) = 0$$

Asymptotic theory cont.

Define operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ as

$$\mathcal{L}_0 = \nu^2 \partial^2_{yy} + (m - y)\partial_y, \tag{7}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu f(y) x^{\frac{\theta}{2}} \partial^2_{xy} - \sqrt{2}\nu\Lambda(t,x,y)\partial_y, \tag{8}$$

$$\mathcal{L}_2 = \partial_t + \frac{1}{2} f(y)^2 x^\theta \partial^2_{xx} + r(x\partial_x - \cdot). \tag{9}$$

then, the PDE (6) can be written as

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P^\epsilon = 0 \tag{10}$$

Asymptotic Expansion

Expand P^ϵ in powers of $\sqrt{\epsilon}$:

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \cdots \tag{11}$$

Here, the choice of the power unit $\sqrt{\epsilon}$ in the power series expansion was determined by the method of matching coefficient.

Asymptotic Expansion cont.

Substituting the PDE (10),

$$\frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots = 0, \tag{12}$$

which holds for arbitrary $\epsilon > 0$.

Asymptotic theory cont.

Lemma 3.1

If solution to the Poisson equation

$$\mathcal{L}_0\chi(y) + \psi(y) = 0 \tag{13}$$

exists, then the following centering (solvability) condition must satisfy $<\psi>= 0$, where $<\cdot>$ is the expectation with respect to the invariant distribution of Y_t. If then, solutions of (13) are given by the form

$$\chi(y) = \int_0^t E^y[\psi(Y_t)]\, dt + \text{constant}. \tag{14}$$

Note:

$$\langle\psi\rangle = \int_{-\infty}^{\infty} \psi(y)f(y)dy, \;\; f(y) = \frac{1}{\sqrt{2\pi\nu^2}}\exp\left(-\frac{(y-m)^2}{2\nu^2}\right)$$

Asymptotic Expansion cont.

From the asymptotic expansion (12) $1/\epsilon$ order, we first have

$$\mathcal{L}_0 P_0 = 0. \tag{15}$$

Solving this equation yields

$$P_0(t,x,y) = c_1(t,x)\int_0^y e^{\frac{(m-z)^2}{2\nu^2}}\,dz + c_2(t,x)$$

for some functions c_1 and c_2 independent of y.

- $c_1 = 0$ is required.
- $P_0(t,x,y)$ must be a function of only t and x

$$P_0 = P_0(t,x).$$

Asymptotic Expansion cont.

- From the expansion (12) $1/\sqrt{\epsilon}$ order ,

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$$

- Known $\mathcal{L}_1 P_0 = 0$
- Get $\mathcal{L}_0 P_1 = 0$

$$P_1 = P_1(t, x) \tag{16}$$

Asymptotic Expansion P_0

Theorem 3.2

The leading term $P_0(t,x)$ is given by the solution of the PDE

$$\frac{\partial P_1}{\partial t} + \frac{1}{2} < f^2 > x^{\theta} \frac{\partial^2 P_1}{\partial x^2} + r(x \frac{\partial P_1}{\partial x} - P_1) = 0 \quad (17)$$

with the terminal condition $P_0(T,x) = h(x)$.

Proof of Theorem 3.2

Proof

From the expansion (12), the PDE

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0 \tag{18}$$

Since $\mathcal{L}_1 P_1 = 0$, then

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0 \tag{19}$$

which is a Poisson equation.

Proof of Theorem 3.2 cont.

From Lemma 3.1 with $\psi = \mathcal{L}_2 P_0$, $P_0(t,x)$ has to satisfy the centering condition

$$<\mathcal{L}_2> P_0 = 0 \tag{20}$$

with the terminal condition $P_0(T,x) = h(x)$, where

$$<\mathcal{L}_2> \; = \; \partial_t + \frac{1}{2} <f^2> x^\theta \partial_{xx}^2 + r(x\partial_x - \cdot).$$

Thus P_0 solves the PDE (17). $\square$

Asymptotic Expansion P_1 cont.

Theorem 3.3

The first correction $P_1(t,x)$ is given by the solution of the PDE

$$\frac{\partial P_1}{\partial t} + \frac{1}{2} < f^2 > x^{\theta} \frac{\partial^2 P_1}{\partial x^2} + r\left(x\frac{\partial P_1}{\partial x} - P_1\right) = V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right) + V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} \tag{21}$$

with the final condition $P_1(T,x) = 0$, where V_3 and V_2 are given by (22) and (23), respectively.

Asymptotic Expansion P_1

For convenience,

$$V_3(x;\theta) = \frac{\nu}{\sqrt{2}} \rho x^{\frac{\theta-2}{2}} < f\psi_y >, \tag{22}$$

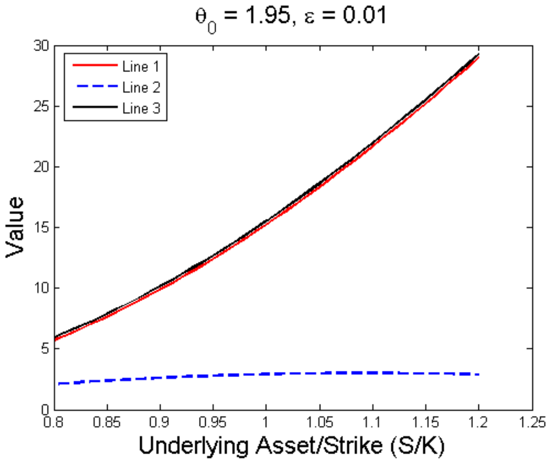
$$V_2(x;\Lambda;\theta) = \frac{\nu}{\sqrt{2}} \left(\rho x^{\frac{\theta}{2}} < f\psi_{xy} > - < \Lambda\psi_y > \right), \tag{23}$$

where $\psi(t,x,y)$ is solution of the Poisson equation

$$\mathcal{L}_0\psi = \nu^2\psi_{yy} + (m-y)\psi_y = \left(f^2 - < f^2 >\right) x^{\theta-2}. \tag{24}$$

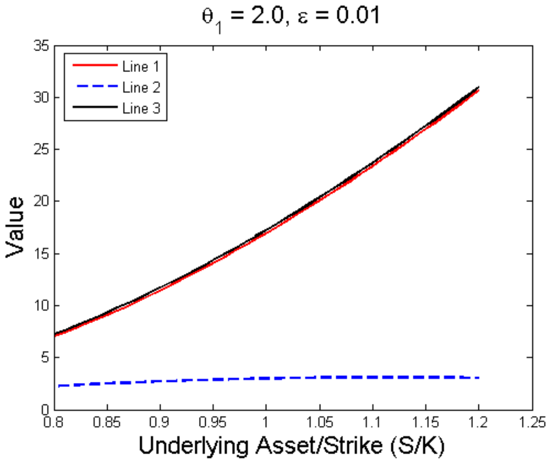
Outline

$\theta = 1.95$ and $\epsilon = 0.01$



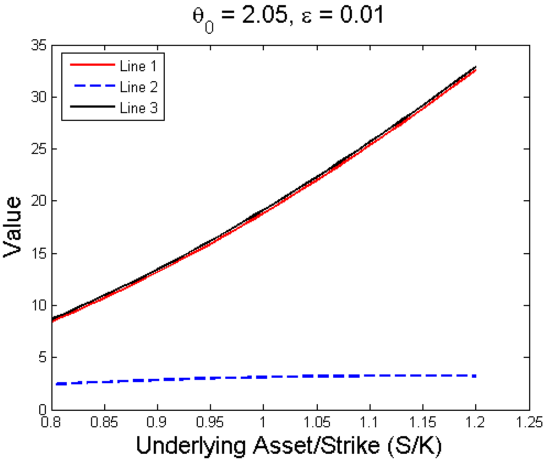
Line 1 $= P_0$, Line 2 $= P_1$, Line 3 $= P_0 + \sqrt{\epsilon} P_1$

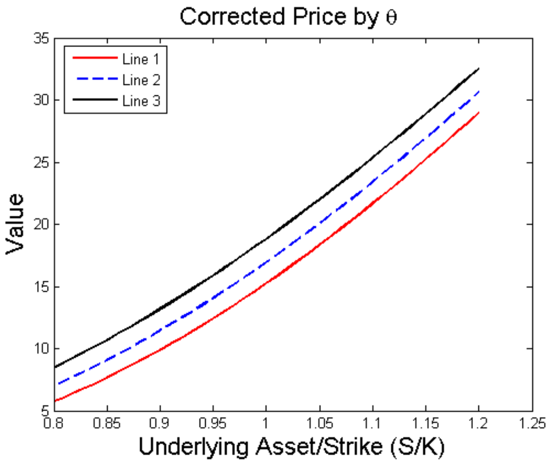
$\theta = 2.00$ and $\epsilon = 0.01$



$P_0(t,x)$ with $K = 100$, $T = 1$, $\overline{\sigma} = 0.165$, and $r = 0.02$. It is computed with terminal condition $h(x) = (x - K)^+$.

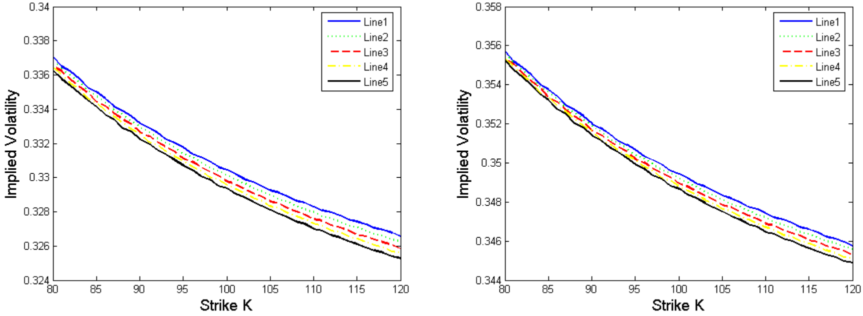
$\theta = 2.05$ and $\epsilon = 0.01$





Line 1 : $\theta = 1.95$, Line 2 : $\theta = 1.95$, Line 3 : $\theta = 2.05$

Dynamics of Implied Volatility ($\theta = 1.9$ and $\theta = 1.925$)

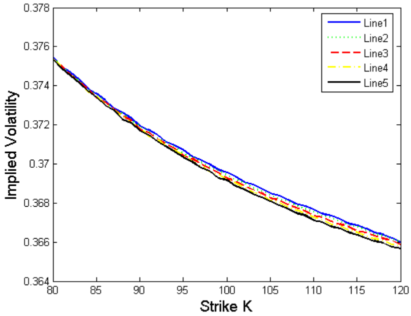


(a) $\theta = 1.9$

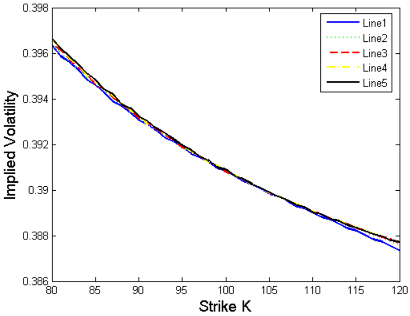
(b) $\theta = 1.925$

Line 1 : $X_0 = 90$, Line 2 : $X_0 = 95$, Line 3 : $X_0 = 100$,
Line 4 : $X_0 = 105$, Line 5 : $X_0 = 110$

Dynamics of Implied Volatility ($\theta = 1.95$ and $\theta = 1.975$)

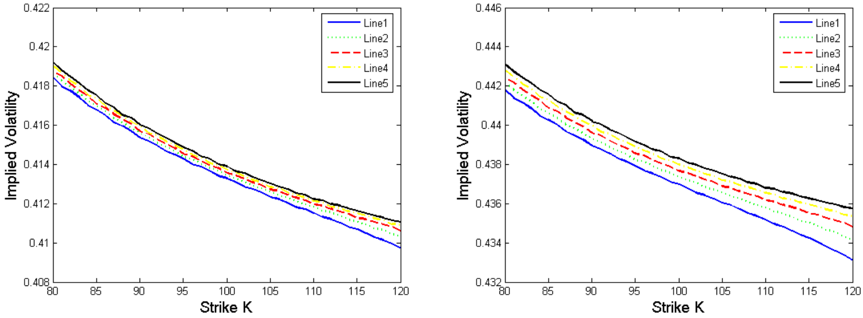


(c) $\theta = 1.95$



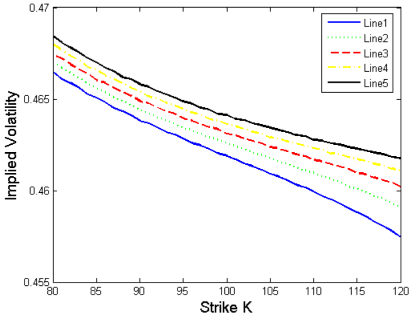
(d) $\theta = 1.975$

Dynamics of Implied Volatility ($\theta = 2.00$ and $\theta = 2.025$)

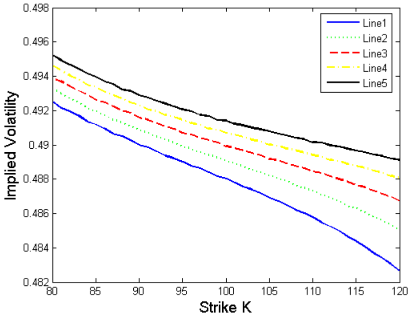


(e) $\theta = 2.0$ (f) $\theta = 2.025$

Dynamics of Implied Volatility ($\theta = 2.05$ and $\theta = 2.075$)

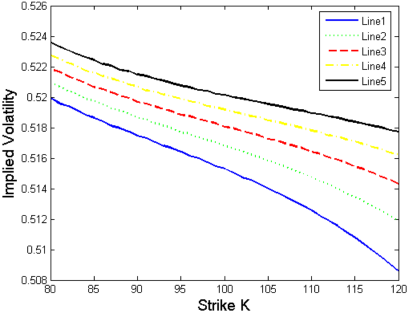


(g) $\theta = 2.05$



(h) $\theta = 2.075$

Dynamics of Implied Volatility ($\theta = 2.1$)



(i) $\theta = 2.1$

Remark

- Implied volatility curve move from left to right for $\theta \geq 1.975$.
- For $\theta \geq 2$, The implied volatility curve seems to be skew, unlike CEV model.

Outline

Conclusion

Conclusion

- Corrected Price (A new hybrid model)
- Right dynamics of Implied Volatility
- Stability of Hedging
- Still ongoing research(Fitting to Market Data)

Outline

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Thank you for your attention!