Exact Sampling of Jump-Diffusion Processes

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Jump-Diffusion Processes

- Ubiquitous in finance and economics
  - Price models: equity, commodity, rates, energy, FX
  - Default timing models (jump component)
- We develop a method for the **exact sampling** of a jump-diffusion process with state-dependent drift, volatility, jump intensity, and jump size
  - Leads to **unbiased** simulation estimators of derivative prices, risk measures, and other quantities
- The method extends an innovative acceptance/rejection scheme developed by Beskos & Roberts (2005, AAP) and Chen (2009) for diffusions

Jump-Diffusion

- Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- For suitable functions μ and σ, consider the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t$$

where W is a standard Brownian motion and J is a jump process

$$J_t = \sum_{n \leq N_t} c(X_{T_n^-}, Z_n)$$

  - N is a counting process with event times (T_n) and intensity $\lambda_t = \Lambda(X_{t-})$ for a suitable function Λ
  - (Z_n) is a sequence of mark variables valued in E
  - $c : D_X \times E \to D_X$ determines the jump magnitudes
- J is self-exciting; dependence between jump sizes and frequency

Jump-Diffusion

- For a suitable function f and a horizon T we wish to calculate

$$\mathbb{E}\{f(X_T, (J_t)_{t \leq T})\}$$

  - Price of a derivative written on X_T
  - Price of a credit derivative written on J

- Alternative approaches
  - Analytical solutions: rare; e.g. Merton and Kou models
  - Semi-analytical transform approaches: AJDs, LQJDs, Lévy
  - PIDE approaches: fewer restrictions
  - Monte Carlo simulation: perhaps the widest scope

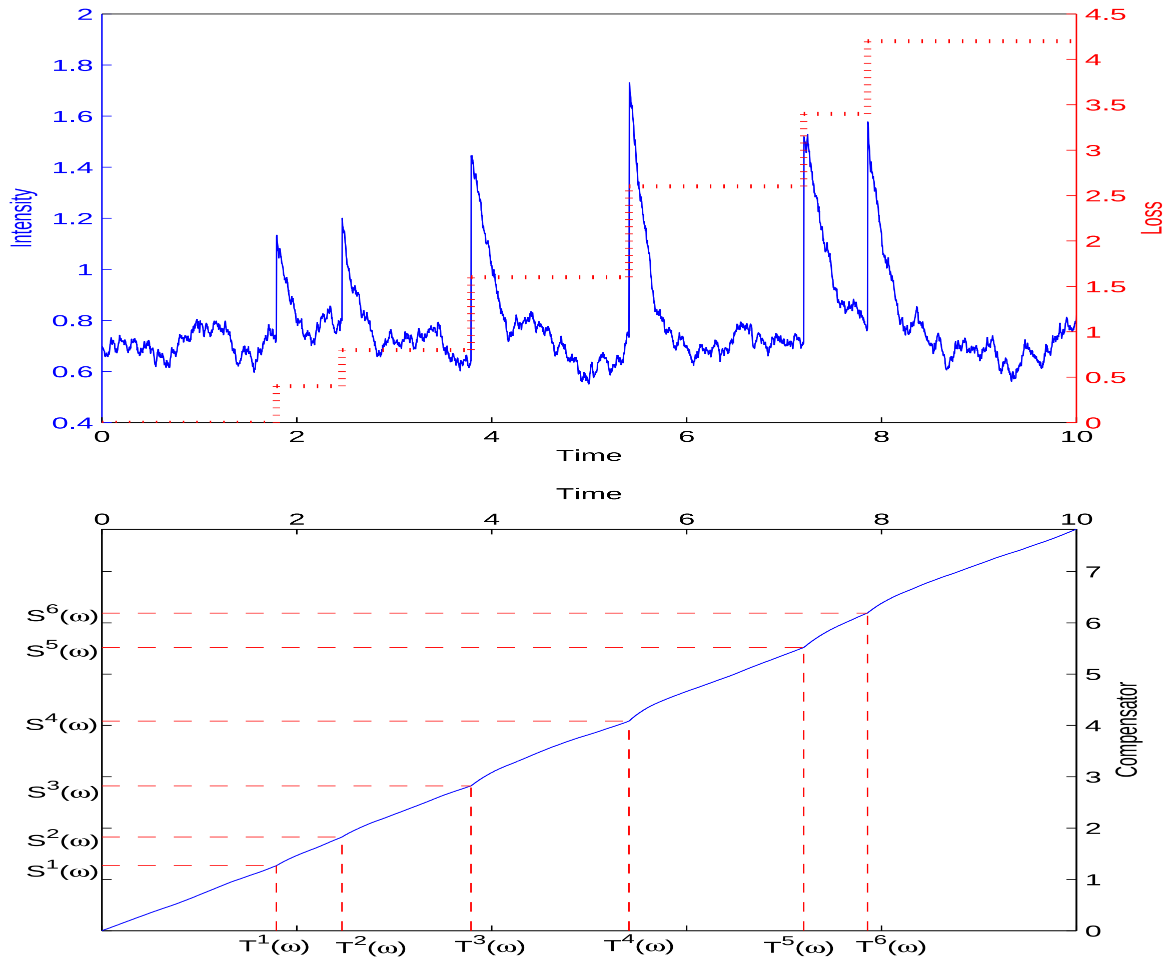
Discretizing the Jump-Diffusion

Standard approach to simulating X

- X is approximated on a discrete-time grid
  - Euler or higher order scheme for diffusion component
  - Thinning or time-scaling scheme for jump times T_n
- Simulation estimator is biased
  - Magnitude of the bias? Confidence intervals?
  - Convergence of scheme?
  - Allocation of computational budget?

Discretizing the Jump-Diffusion

Time-scaling for jumps: $T_n \stackrel{d}{=} \inf\{t : \int_0^t \Lambda(X_s)ds \geq \mathcal{E}_1 + \cdots + \mathcal{E}_n\}$



Exact Sampling

- We provide an exact sampling scheme for X that avoids discretization entirely, and leads to unbiased simulation estimators
- First step: transform X into a unit-diffusion SDE
  - Lamperti transform $F(x) = \int_{X_0}^{x} \frac{1}{\sigma(u)} du$
  - Then $Y_t = F(X_t)$ solves

$$dY_t = \alpha(Y_t)dt + dW_t + dJ_t^Y$$

where

$$\alpha(y) = \frac{\mu(F^{-1}(y))}{\sigma(F^{-1}(y))} - \frac{1}{2}\sigma'(F^{-1}(y))$$

$$J_t^Y = \sum_{n \leq N_t} \Delta(Y_{T_n^-}, Z_n)$$

for $\Delta(y,z) = F(F^{-1}(y) + c(F^{-1}(y), z)) - y$

Exact Sampling

Assumptions

1. $\sigma(x)$ is bounded away from 0
2. $\mu(x)$ and $\Lambda(x)$ are continuously differentiable and $\sigma(x)$ is twice continuously differentiable.
3. Conditions on $\alpha(y)$ and $c(x,z)$ guaranteeing that Y does not reach the boundaries in finite time (known).

Acceptance/Rejection Scheme

- The exact method uses an A/R scheme
- Suppose we want to sample from a density $f(y)$ and there is another density $g(y)$ and a constant $c > 0$ such that

$$c \cdot \frac{f(y)}{g(y)} \leq 1$$

- A/R scheme
  1. Draw a sample Y from g
  2. Draw a Bernoulli variable I with success probability $c \cdot \frac{f(Y)}{g(Y)}$
  3. Accept Y as a sample from f if $I = 1$

A/R for Continuous Y

Beskos & Roberts (2005, AAP)

- We wish to sample Y_T using the A/R scheme
- Under Novikov and additional boundedness conditions,

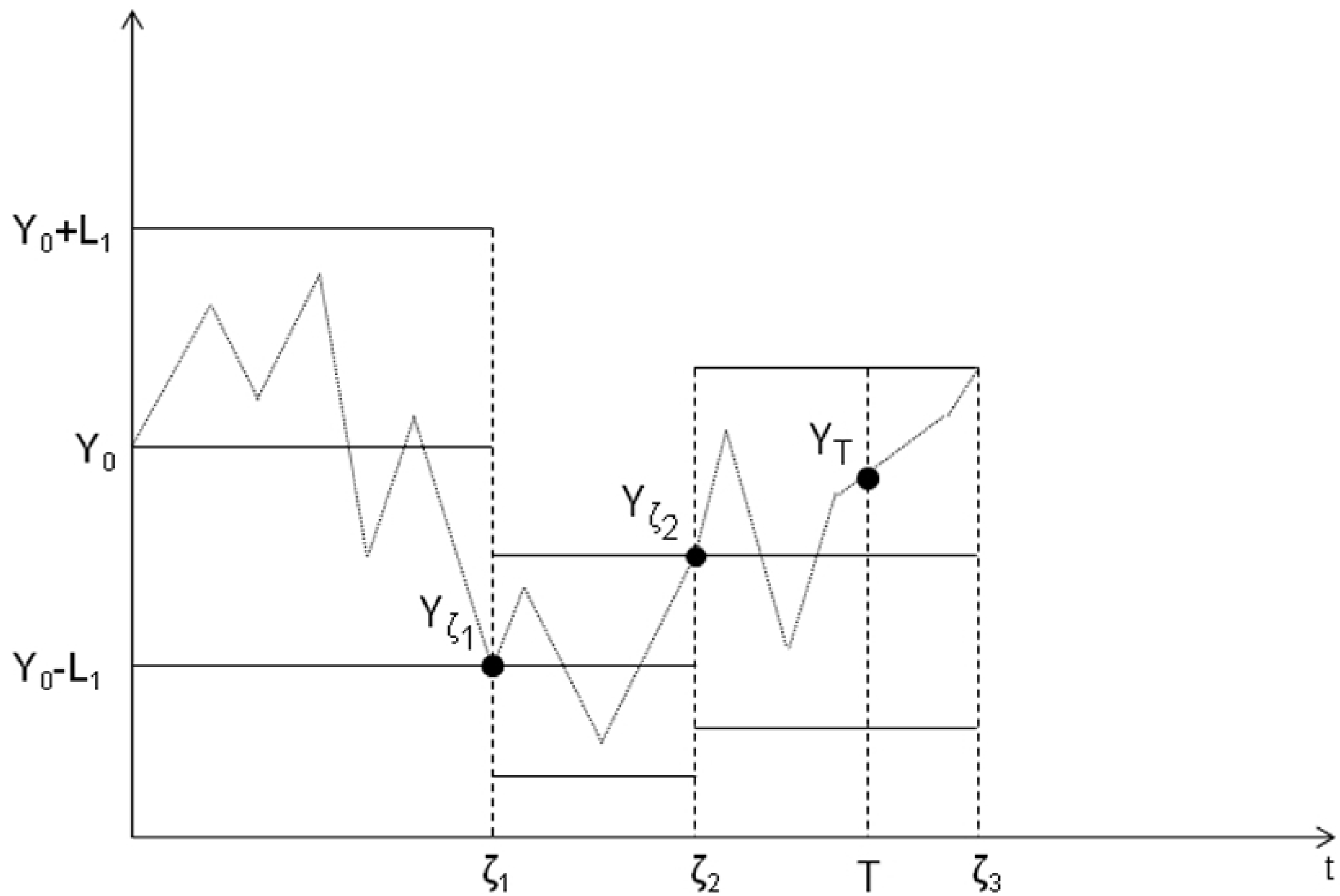
$$\frac{f_{Y_T}(y)}{g(y)} \propto \mathbb{E}\left[\exp\left(-\int_0^T \phi(W_s)ds\right)\Big|\, W_T = y\right] =: H(y)$$

  where $\phi = (\alpha' + \alpha^2)/2$ and $g(y)$ is a proposal density

- For the A/R step, note that $H(y) = \mathbb{P}(M_T = 0 \,|\, W_T = y)$ where M is a doubly-stochastic Poisson process with intensity $\phi(W_s)$
  - The Bernoulli indicator $\{I = 1\} = \{M_T = 0\}$ can be generated by sampling M_T given $W_T = Y$ with Y drawn from g
  - If $\phi(x) \leq \pi$, then this can be done by thinning a Poisson process with intensity π (requires sampling from BB)

A/R for Continuous Y

Localization: Chen (2009)



A/R for Continuous Y

Generating the first piece of Y

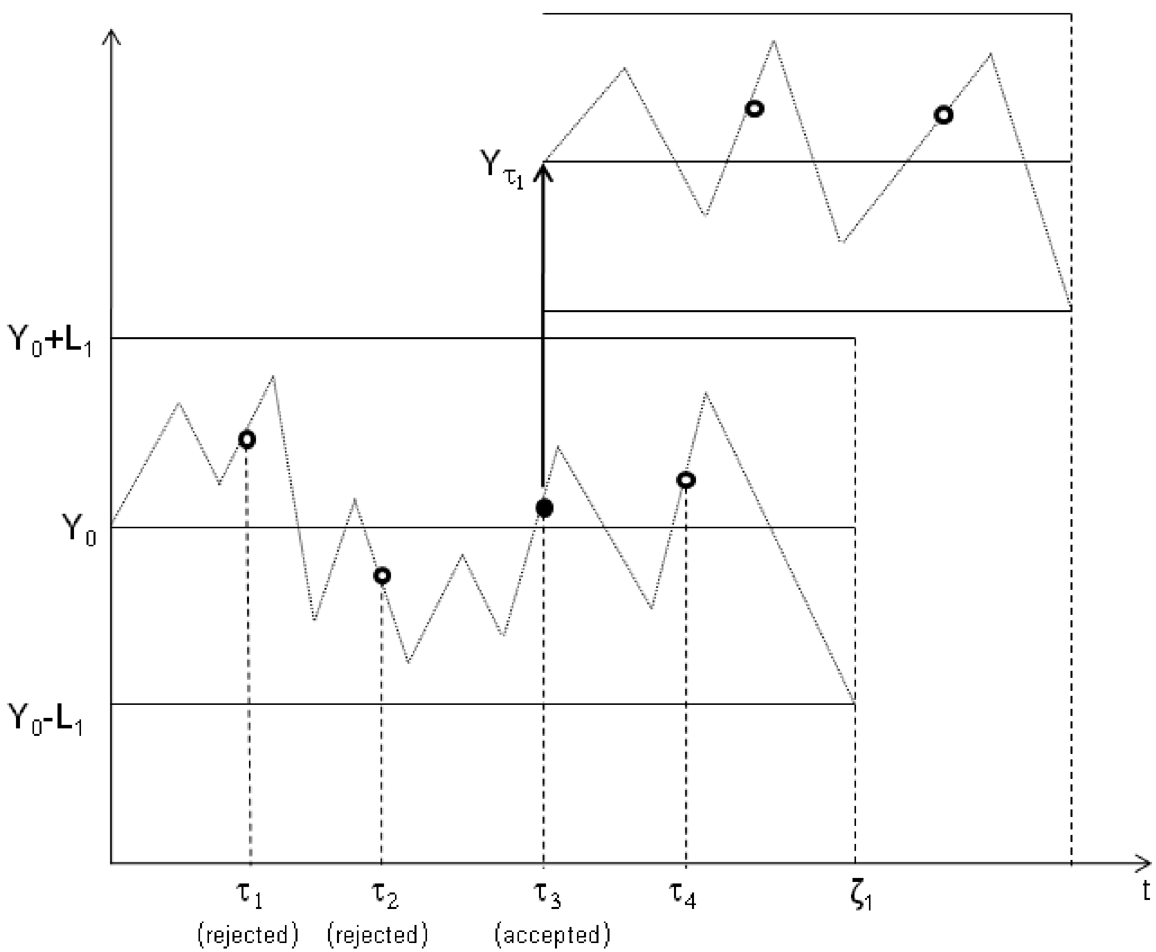
- Target exit time $\zeta_1 = \inf\{t \geq 0 : |Y_t - Y_0| \geq L\}$ for $L > 0$
- Proposal exit time $\tau = \inf\{t \geq 0 : |W_t| \geq L\}$
- The LR between $(\zeta_1, Y_1 - Y_0)$ and (τ, W_τ) is proportional to

$$\mathbb{E}\left[\exp\left(-\int_0^\tau \phi(Y_0 + W_s)ds\right) \Big| \tau, W_\tau\right]$$

- Because of the continuity assumptions, $\phi(Y_0 + W_s)$ is bounded and thinning can always be used in the acceptance test
  - Need to sample from Brownian meander
- The density of τ is known and can be sampled from using the method of Burq & Jones (2006)

A/R for Jump-Diffusion Y

Introducing jumps



A/R for Jump-Diffusion Y

Generating the first piece of Y

- Sample τ as before using a bound L. Suppose $\tau \leq T$.
- Sample candidate jump times $(\sigma_1, \dots, \sigma_n)$ of Y during $[0, \tau]$ from a Poisson process with intensity $\pi \geq \Lambda(F^{-1}(Y_0 + y))$, $y \in [-L, L]$
- Sample $(W_{\sigma_1}, \dots, W_{\sigma_n}, W_\tau)$ from a Brownian meander
- Perform acceptance tests for the σ_i by drawing Bernoulli variables with success probabilities $\Lambda(F^{-1}(Y_0 + W_{\sigma_i}))/\pi$
- Perform acceptance test for $(\zeta_1, Y_{\sigma_1} - Y_0, \dots, Y_{\sigma_k^-} - Y_0)$ given the proposal $(\tau, W_{\sigma_1}, \dots, W_{\sigma_k})$, where σ_k is the first candidate jump time of Y accepted in the previous step
- If the skeleton is accepted, draw mark Z_1 and compute $Y_{T_1} = Y_{\sigma_k^-} + \Delta(Y_{\sigma_k^-}, Z_1)$

A/R for Jump-Diffusion Y

Likelihood ratio

- The LR for the last acceptance test is proportional to

$$e^{A(Y_0+W_{\sigma_k})}\mathbb{E}\left[\exp\left(-\int_0^{\sigma_k}\phi(Y_0+W_u)du\right)\,\middle|\,\tau,W_{\sigma_1},\ldots,W_{\sigma_k}\right]$$

  where $A(x)=\int_0^x \alpha(u)du$ and $\phi=(\alpha'+\alpha^2)/2$

- Generate Bernoulli indicator by generating the jump times of a doubly-stochastic Poisson process with intensity $\phi(Y_0+W_u)$
  - Thinning applies
  - Sample from Brownian meander

Numerical examples

- Jump-extended CEV model of Carr & Linetsky (2006):

$$dX_t = (r + \Lambda(X_t))X_t dt + \sigma(X_t)dW_t + dJ_t$$

  where $X_0 > 0$ and for $a > 0$, $b \geq 0$, $c \geq \frac{1}{2}$ and $\beta < 0$

  - $\Lambda(x) = b + ca^2x^{2\beta}$ is the jump intensity
  - $\sigma(x) = ax^{\beta+1}$ is the volatility
  - $c(x, z) = -xz$ for $z \in (0, 1)$ is the jump size

- The firm defaults at the first jump time T_1 of J

- The default intensity $\lambda = \Lambda(X)$ is unbounded
  - Violates boundedness hypothesis of thinning scheme for jumps (Glasserman & Merener (2003), Casella & Roberts (2010))
  - Convergence order of discretization scheme unknown

Numerical examples

- We are interested in X during $[0, T \wedge T_1]$ for some $T > 0$
- The target functional takes the form

$$f(X_T, (J_t)_{t \leq T}) = h_1(X_T) 1_{\{J_T = 0\}} + h_2(X_T) 1_{\{J_T \neq 0\}}$$

  Examples

  - Probability of survival to T: $h_1(x) = 1$ and $h_2(x) = 0$
  - European put with strike K and maturity T: $h_1(x) = e^{-rT}(K - x)^+$ and $h_2(x) = Ke^{-rT}$

- Carr & Linetsky (2006) provide analytical solutions to these and other quantities

Numerical examples

- We estimate the price of a European put on X
- We consider the RMSE $= \sqrt{\text{Bias}^2 + \text{SE}^2}$ for
  - The exact method, for which the bias is 0
  - The discretization method (Euler plus time-scaling for jumps)
    - The number of time steps is equal to the square root of the number of trials (Duffie & Glynn (1995))
    - The bias is estimated using 10 million trials
- Matlab implementation (favors discretization)

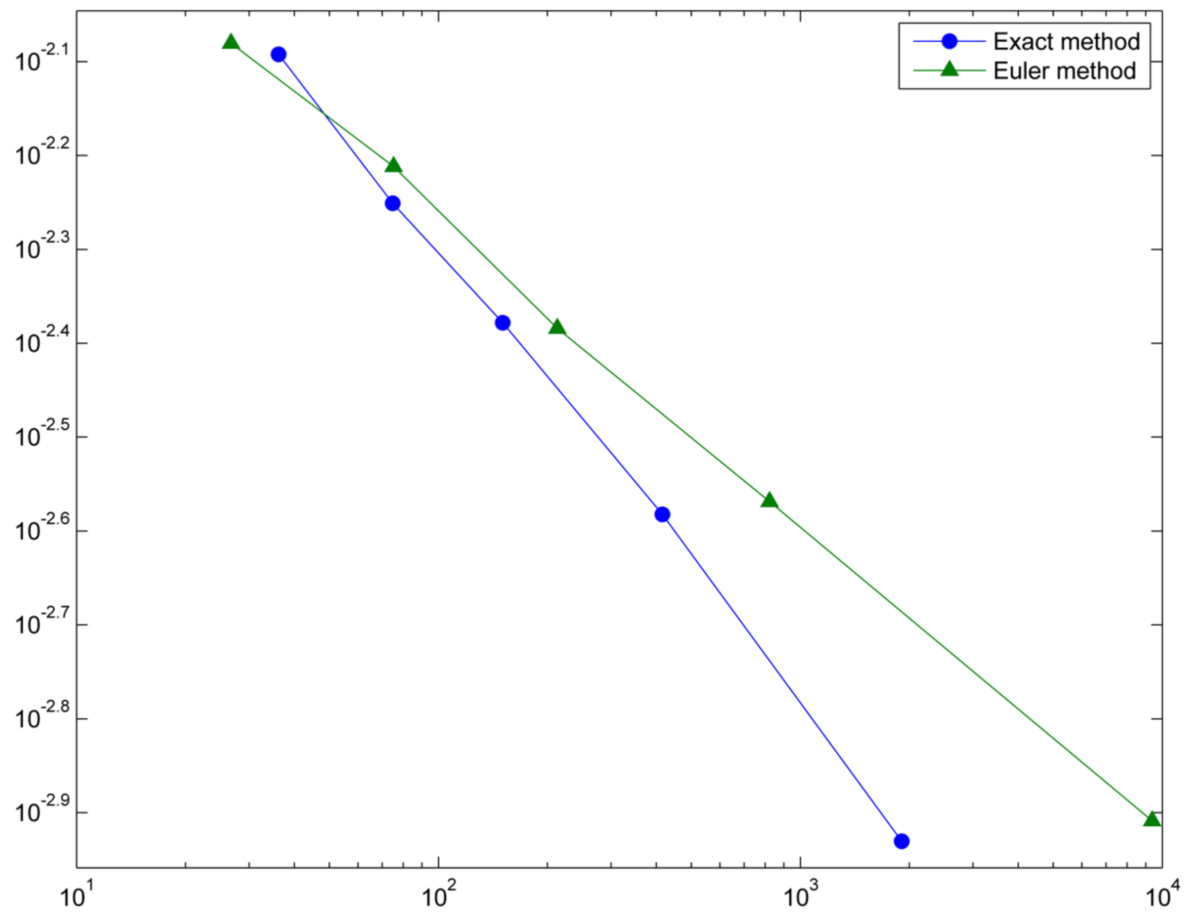
Numerical examples

$X_0 = 50, \beta = -1, r = 0.05, a = 50/4, b = 0, c = 0.5$, $T = 1$, strike 5, analytical value 0.1491

Method	Trials	Steps	Value	Bias	SE	RMSE	Time (sec)
Exact	10K	N/A	0.1417	0	0.00809	0.00809	36.19
Exact	20K	N/A	0.1363	0	0.00561	0.00561	74.75
Exact	40K	N/A	0.1522	0	0.00419	0.00419	150.52
Exact	100K	N/A	0.1487	0	0.00262	0.00262	416.07
Exact	500K	N/A	0.1495	0	0.00117	0.00117	1905.71
Euler	10K	100	0.1417	0.0019	0.00809	0.00831	26.72
Euler	20K	140	0.1496	0.0018	0.00587	0.00624	75.19
Euler	40K	200	0.1422	0.0008	0.00405	0.00413	215.75
Euler	100K	310	0.1531	0.0005	0.00265	0.00271	822.07
Euler	500K	707	0.1478	0.0004	0.00117	0.00123	9373.81

Numerical Examples

Convergence of RMSEs (log-log plot), strike $K = 5$



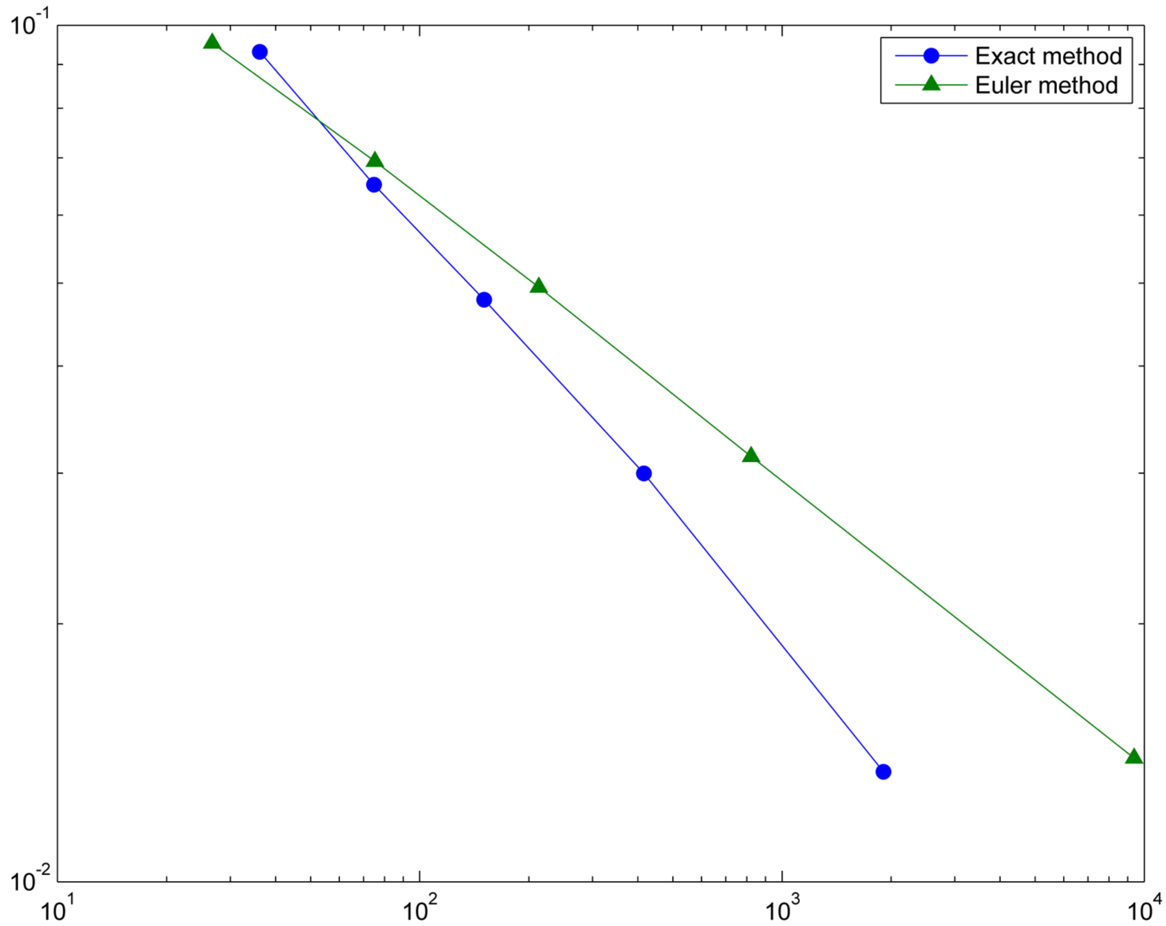
Numerical examples

$X_0 = 50, \beta = -1, r = 0.05, a = 50/4, b = 0, c = 0.5$, $T = 1$, strike 50, analytical value 4.4118

Method	Trials	Steps	Value	Bias	SE	RMSE	Time (sec)
Exact	10K	N/A	4.3773	0	0.09309	0.09309	36.19
Exact	20K	N/A	4.2248	0	0.06512	0.06512	74.75
Exact	40K	N/A	4.4464	0	0.04781	0.04781	150.52
Exact	100K	N/A	4.4123	0	0.02996	0.02996	416.07
Exact	500K	N/A	4.4214	0	0.01344	0.01344	1905.71
Euler	10K	100	4.3198	0.0185	0.09345	0.09527	26.72
Euler	20K	140	4.451	0.0163	0.06742	0.06936	75.19
Euler	40K	200	4.3876	0.0157	0.04691	0.04946	215.75
Euler	100K	310	4.4464	0.0081	0.03031	0.03136	822.07
Euler	500K	707	4.3914	0.0039	0.01338	0.01394	9373.81

Numerical Examples

Convergence of RMSEs (log-log plot), strike $K = 50$



Extensions

- The target functional can take the form

$$\mathbb{E}\{f((X_t)_{t\in S},(J_t)_{t\leq T})\}$$

for a discrete set S of times $t \in [0, T]$

  - Treatment of certain path-dependent payoffs

- The intensity can take the form

$$\lambda_t = \Lambda(X_{t-}, J_{t-}, t)$$

Conclusions

- We develop a method for the exact sampling of a one-dimensional jump-diffusion process with state-dependent drift, volatility, jump intensity and jump size
  - Only mild conditions on the coefficients are required
- Numerical experiments indicate the advantages of the method over a conventional discretization scheme
- Future research
  - Efficiency: choice of localization bound
  - Extension to multiple dimensions: stochastic volatility with state-dependent jumps