Credit Modelling by Particle Systems and Stochastic PDEs

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joint work with Ben Hambly

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25 June 2010, Bachelier Finance 6th World Congress, Toronto

Outline

Motivation and Introduction

- In Finance: credit market needs an improved mathematical model for pricing and doing risk management to cope with the increasing complexity of the credit product traded.
- In Mathematics: interested in proving the existence and uniqueness of the solution to a class of SPDEs with some boundary condition.
- Basket credit modelling – correlation structure – bottom-up approach and top-down approach.
- Bottom-up approach: Copulas and Conditionally Independent Factor models, which do not have any time-related property.
- We apply the particle system and SPDE techniques to basket credit modelling under the structural model framework.

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Contribution

- Develop a dynamic multi-dimensional structural model;
- Use particle representation with absorption for asset values of firms and assume interactive dynamics for the particles;
- Derive an evolution equation satisfied by the limit empirical measure of the particle system;
- Derive an SPDE for the density of the limit measure;
- Give an explicit formula for the default proportion at any time;
- Give estimates for the limit measure behaviour near the absorbing boundary;
- Prove the uniqueness of the solution to the SPDE in some cases.

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Model Description (1)

- n: number of firms in the credit basket;
- A_t^i: the asset value of the i-th firm at time t;
- B^i: the constant default boundary of the i-th firm;
- T_0^i: the stopping time when the i-th firm defaults, i.e.,

$$T_0^i := \inf\{t : A_t^i = B^i\};$$

- $a_{n,t}$: the default proportion in the credit basket by time t, i.e.,

$$a_{n,t} := \frac{1}{n} \sum_{i=1}^{n} 1_{\{T_0^i \leq t\}}.$$

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Model Description (2)

- The dynamics of the i-th ($i = 1, 2, ..., n$) firm's asset value A_t^i:

$$\begin{cases} \frac{dA_t^i}{A_t^i} &= \mu(t, A_t^i)dt + \sqrt{1-\rho^2(t, A_t^i)}\sigma dW_t^i + \rho(t, A_t^i)\sigma dM_t, \ t < T_0^i, \\ A_t^i &= B^i, \quad t \geq T_0^i, \\ A_0^i &= b^i > B^i, \end{cases}$$

- W_t^i: idiosyncratic noise;
- M_t: common factor;
- W_t^i and M_t are standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $d\langle W_t^i, M_t \rangle = 0$, $\forall i$ and $d\langle W_t^i, W_t^j \rangle = \delta_{ij} dt$, $\forall i, j$;
- $0 \leq |\rho(t, A_t^i)| \leq 1$, $\forall i = 1, 2, ..., n$, σ is a positive constant.

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Model Description (3)

- Consider the *distance to default* process:

$$X_t^i = \frac{\log A_t^i - \log B^i}{\sigma}.$$

- By Itô's formula, the dynamics of X_t^i:

$$\begin{cases} dX_t^i &= \hat{\mu}(t, X_t^i)dt + \sqrt{1 - \rho^2(t, X_t^i)}dW_t^i + \rho(t, X_t^i)dM_t, \ t < T_0^i, \\ X_t^i &= 0, \quad t \geq T_0^i, \\ X_0^i &= x^i > 0, \end{cases}$$

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Model Description (4)

- The empirical measure for the particle system $\{X_t^i\}$: $\nu_{n,t} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^i}$, which has support $\mathbb{R}^+ \cup \{0\}$.
- The limit empirical measure:

$$\nu_t = \lim_{n\to+\infty} \nu_{n,t} = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^i}.$$

- Existence of ν_t?

Theorem

Under the assumption that $\{X_0^i\}$ is a family of exchangeable random variables which are independent of $\{W^i\}$ and M, there exists a $C_{\mathcal{P}(\mathbb{R})}[0,\infty)$-valued random variable ν_t such that

$$\nu_t = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^i}, \text{ a.s..}$$

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Main Result for the Limit Empirical Measure

Theorem

$$\nu_t = \nu_t^+ + a_t\delta_0,$$

where for any appropriate test function ϕ*,*

$$\begin{cases} \langle \phi, \nu_t^+ \rangle &= \langle \phi, \nu_0^+ \rangle + \int_0^t \langle \hat{\mu}(s,x)\phi'(x) + \frac{1}{2}\phi''(x), \nu_s^+ \rangle ds \\ & \quad + \int_0^t \langle \rho(s,x)\phi'(x), \nu_s^+ \rangle dM_s, \\ \nu_0^+ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{x^i}. \end{cases}$$

Also $\lim_{\varepsilon\downarrow 0} \frac{\nu_t^+((0,\varepsilon])}{\varepsilon} = 0,$ *a.s. and moreover,*

$$a_t = \lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{T_0^i \le t\}} = \lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_t^i = 0\}}$$
$$= 1 - \nu_t^+((0,+\infty)).$$

Limit Measure Behaviour Near Boundary

- Using Iyengar (1985)'s formula for the joint distribution of two correlated Brownian motions and their first passage times, we have the following estimate for the second moment of the limit measure:

Proposition

Assume that $\{X_0^i\}$ is an exchangeable family whose distribution is bounded away from zero, i.e. with probability 1 we have

$$X_0^i \geq C_B > 0, \ \forall i.$$

Then there exist $\tilde{\varepsilon}_0 > 0$ only depending on ρ and the bound C_B and $\beta > 0$ such that for all $\varepsilon < \tilde{\varepsilon}_0$ we have

$$\mathbb{E}[(\nu_t^+((0, \varepsilon]))^2] \leq K_T \varepsilon^{3+\beta},$$

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Density of the Limit Empirical Measure

- Density existence?

Theorem

If ν_t^+ is a measure-valued solution of the evolution equation, and $\nu_0^+ \in L^2(\mathbb{R}^+)$, then $\nu_t^+ \in L^2(\mathbb{R}^+)$, a.s. and $\mathbb{E}||\nu_t^+||_0^2 < \infty, \forall t \geq 0$.

- Write $\nu_t^+ = v(t,x)dx, \; x \in \mathbb{R}^+$.
- The SPDE satisfied by the density:

$$\begin{aligned} dv(t,x) = & -\frac{\partial}{\partial x}(\hat{\mu}(t,x)v(t,x))dt + \frac{1}{2}v_{xx}(t,x)dt \\ & -\frac{\partial}{\partial x}(\rho(t,x)v(t,x))dM_t, \quad x \in \mathbb{R}^+. \end{aligned}$$

- Boundary condition:

$$v(t,0) = 0, \; \forall t.$$

- Limit Default Proportion:

$$a_t = 1 - \int_0^{+\infty} v(t,x)dx (= \frac{1}{2}\int_0^t v_x(s,0+)ds).$$

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- Write $\nu_t^+ = v(t,x)dx, \ x \in \mathbb{R}^+$.
- The SPDE satisfied by the density:

$$dv(t,x) = -\frac{\partial}{\partial x}(\hat{\mu}(t,x)v(t,x))dt + \frac{1}{2}v_{xx}(t,x)dt - \frac{\partial}{\partial x}(\rho(t,x)v(t,x))dM_t, \quad x \in \mathbb{R}^+.$$

- Boundary condition:

$$v(t,0) = 0, \ \forall t.$$

- Limit Default Proportion:

$$a_t = 1 - \int_0^{+\infty} v(t,x)dx (= \frac{1}{2}\int_0^t v_x(s,0+)ds).$$

Existence and Uniqueness of the SPDE

- Existence: have shown the existence of the solution in a constructive way by giving a particle representation to the SPDE – the solution is the L^2-density of the limit empirical measure of the particle system $\{X_t^i\}$.
- Uniqueness: (for constant correlation ρ)

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Suppose that $\nu_0^+ \in H_0$. Then the evolution equation has at most one measure-valued solution.

- Krylov's theorem:

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Under appropriate hypotheses, the SPDE has a unique solution $v(t, x)$ such that $v \in L_2(\Omega \times (0, T), \mathcal{B}(\Omega \times \mathbb{R}^+), H^1(\mathbb{R}^+))$ and is such that $\min(x, 1)v_{xx} \in L_2(\Omega \times (0, T), \mathcal{B}(\Omega \times \mathbb{R}^+), L^2(\mathbb{R}^+))$*, where $H^1(\mathbb{R}^+)$ is the usual Sobolev space.*

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Data and Calibration (1)

- By Christoph Reisinger (Oxford) and Helen Haworth (Credit Suisse).
- Consider the simplest version of our model:

$$dv = -\frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)v_x dt + \frac{1}{2}v_{xx}dt - \sqrt{\rho}v_x dM_t,$$

with the initial condition $v(0,x) = v_0(x)$ and the boundary condition $v(t,0) = 0$.

Maturity Date	Fixed Coupon (bp)	Traded Spread (bp)	Model Spread (bp)
20/12/2011	30	21	19.6
20/12/2013	40	30	30.7
20/12/2016	50	41	41.0

Table: The fixed coupons, traded spreads and model spreads for the iTraxx Main Series 6 index on February 22, 2007. Parameters used for the model spreads are $r = 0.042$, $\sigma = 0.22$, $R = 0.4$. (Index spreads are correlation (ρ)-independent.)

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Data and Calibration (2)

	5 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	7.19%	7.55%	4.99%	2.14%	-0.71%	-3.48%	-6.17%	-8.78%
3%-6%	41	15.6	55.6	86.4	106.1	116.2	119.5	117.4
6%-9%	10.8	0.7	9.1	25	40.3	54.5	65.2	71.7
9%-12%	5	0	2.2	8.2	18.8	28.6	37.2	45.4
12%-22%	1.8	0	0.2	1.7	4.9	9.8	16.1	22.5
22%-100%	0.9	0	0	0	0.1	0.3	0.7	1.5
	7 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	22.1%	27.45%	19.97%	13.79%	8.31%	3.27%	-1.47%	-6.04%
3%-6%	110	130.6	183.3	202.2	206	201.5	191.6	177.8
6%-9%	32.5	15.3	52.4	80.5	99.1	110.6	116.1	116.9
9%-12%	15	1.8	17.4	37.1	54.3	67.1	76.5	82.7
12%-22%	4.9	0.1	2.3	8.9	19	29.9	39.5	47.9
22%-100%	2	0	0	0.1	0.4	1.1	2.3	4.1

Data and Calibration (3)

- Post credit crisis:

Maturity Date	Fixed Coupon (bp)	Traded Spread (bp)	Model Spread (bp)
20/12/2013	120	215	207
20/12/2015	125	195	195
20/12/2018	130	175	176

Table: The fixed coupons, traded spreads and model spreads for the iTraxx Main Series 10 index on December 5, 2008. Parameters used for the model spreads are $r = 0.033$, $\sigma = 0.136$, $R = 0.4$. (Index spreads are correlation (ρ)-independent.)

Data and Calibration (4)

	5 Year							
Tranche	Market	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
0%-3%	71.5%	81.88%	75.9%	69.56%	63.02%	56.25%	49.16%	41.56%
3%-6%	1576.3	2275.2	1978.5	1743.2	1546.8	1374.6	1222.8	1090.1
6%-9%	811.5	1273.1	1168.2	1079.7	1001.4	931.3	864.6	796.3
9%-12%	506.1	775.7	765.8	748.6	724.7	695.8	663.2	629.1
12%-22%	180.3	307.8	353.3	384.7	405.5	418.1	423.4	420.5
22%-100%	77.9	9.2	16.5	25	34.3	44.5	55.7	68.1
	7 Year							
Tranche	Market	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
0%-3%	72.9%	84.03%	78.98%	73.26%	66.93%	60%	52.41%	44.13%
3%-6%	1473.2	2327.3	1985.7	1715.2	1493.4	1308	1147.8	1001.3
6%-9%	804.2	1344.2	1199	1085.2	988.2	900.7	820.9	747.9
9%-12%	512.4	855.4	808.4	765.3	725.3	684.8	643	600.4
12%-22%	182.6	375.4	401.7	417.6	425.6	427.4	423.1	411.8
22%-100%	75.8	14	22	30.6	39.6	49.3	59.7	71.2

Remark

- Our SPDE model allows us a dynamic view of credit portfolios, therefore, it allows us to price time-related exotic CDO products such as Forward starting CDOs.
- However, it clearly does not do a good job of capturing the tranche spreads accurately-especially after the crisis.
- Question: how to make the senior tranches more risky?
- Answer: to model the underlying asset value by jump-diffusion processes, or more general Lévy processes; or even to add some contagion effect among the firms.

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Conclusion and Future Work

- We have given the explicit description of the particle system's limit empirical measure, thus obtain the exact formula for the limit default proportion at any time.
- We have treated the cases that the underlying common factor is a Brownian motion.
- Future Research Directions:
  jump-diffusion case;
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  model with contagion effect;
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Thank you very much!