Multiscale Stochastic Volatility Models

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Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives

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Price Expansion

P: price of a vanilla European option (to start with)

$$\mathbf{P} = \mathbf{P_0} \quad + \quad \mathbf{v_0}\partial_\sigma \mathbf{P_0} + \mathbf{v_1}\mathbf{D_1}\partial_\sigma \mathbf{P_0} + \mathbf{v_2}\mathbf{D_2}\mathbf{P_0} + \mathbf{v_3}\mathbf{D_1}\mathbf{D_2}\mathbf{P_0}$$
$$\quad + \quad \mathbf{v_4}\partial^{\mathbf{2}}_{\sigma\sigma}\mathbf{P_0} + \cdots$$

$$\mathbf{D_1} = S\frac{\partial}{\partial S}\,(Delta), \quad \mathbf{D_2} = S^2\frac{\partial^2}{\partial S^2}\,(Gamma) \quad \partial_\sigma = \frac{\partial}{\partial \sigma}\,(Vega)\cdots$$

$$\mathbf{v_i} = \mathbf{v_i}(\tau), \text{ payoff independent}, \quad \tau = \text{ time-to-maturity}$$

$\mathbf{P_0}$ is typically a **constant volatility price** $\rightarrow$ closed-form formula

Black-Scholes in Equity (Vasicek or CIR in Fixed Income, Black-Cox in Credit, ...)

Where do we get such an expansion?

What do we expect from it?

Wish List

$$\begin{aligned} P = P_0 \quad &+ \quad v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 \\ &+ \quad v_4 \partial^2_{\sigma\sigma} P_0 + \cdots \end{aligned}$$

- **Accuracy**: the truncated expansion should be a good approximation ($v_i \to 0$ fast enough)

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- **Stability**: the coefficients v's should be stable in time "**short-time tight-fit** vs. **long-time rough fit**"

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$$P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 + v_4 \partial^2_{\sigma\sigma} P_0 + \cdots$$

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Let's look at **calibration** first $\longrightarrow$

Calibration on Implied Volatilities

For vanilla European options we have: $\partial_\sigma \mathbf{P_0} = \tau\bar{\sigma}\mathbf{D_2 P_0}$ so that

$$P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + \frac{v_2}{\bar{\sigma}\tau}\partial_\sigma P_0 + \frac{v_3}{\bar{\sigma}\tau} D_1 \partial_\sigma P_0 + \cdots$$

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For Calls, $P_0 = C_{BS}$ and by direct computation

$$P = C_{BS} + \left\{ v_0 + \frac{v_2}{\bar{\sigma}\tau} + \left(v_1 + \frac{v_3}{\bar{\sigma}\tau} \right) \left(1 - \frac{d_1}{\bar{\sigma}\sqrt{\tau}} \right) \right\} \partial_\sigma C_{BS} + \cdots$$

where $d_1 = \frac{-\mathrm{LM} + (r + \frac{1}{2}\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}$, and $\mathrm{LM} \equiv \log(K/S)$

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Expanding the **implied volatility** $I = \bar{\sigma} + I_1 + \cdots \quad \rightarrow$

$$P \equiv C_{BS}(\bar{\sigma} + I_1 + \cdots) = C_{BS} + I_1 \partial_\sigma C_{BS} + \cdots$$

$$\Longrightarrow I_1 = v_0 + \frac{v_2}{\bar{\sigma}\tau} + \left(v_1 + \frac{v_3}{\bar{\sigma}\tau}\right)\left(1 - \frac{d_1}{\bar{\sigma}\sqrt{\tau}}\right) + \cdots$$

Affine in LMMR: $\quad \mathbf{I} = \mathbf{b} + \mathbf{a}\,\frac{\mathrm{LM}}{\tau} + (\textbf{quartic in LM}) + \cdots$

where the term structure of the v's (τ dependence) is important.

Calibration Examples

Goal: fit

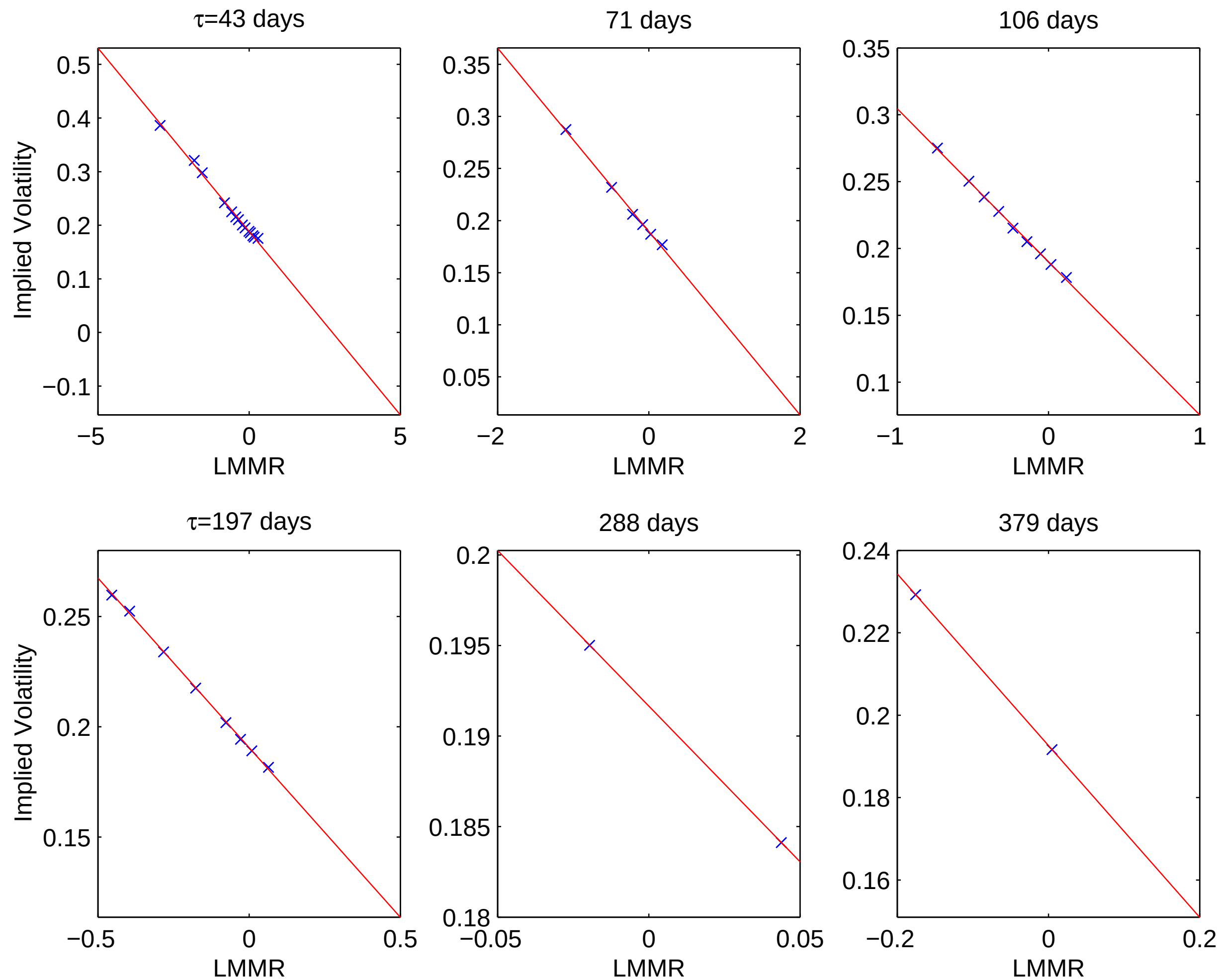
$$\mathbf{I} = \mathbf{b} + \mathbf{a}\,\frac{\mathrm{LM}}{\tau} + (\textbf{quartic in LM}) + \cdots$$

to the observed **implied volatility surface**.

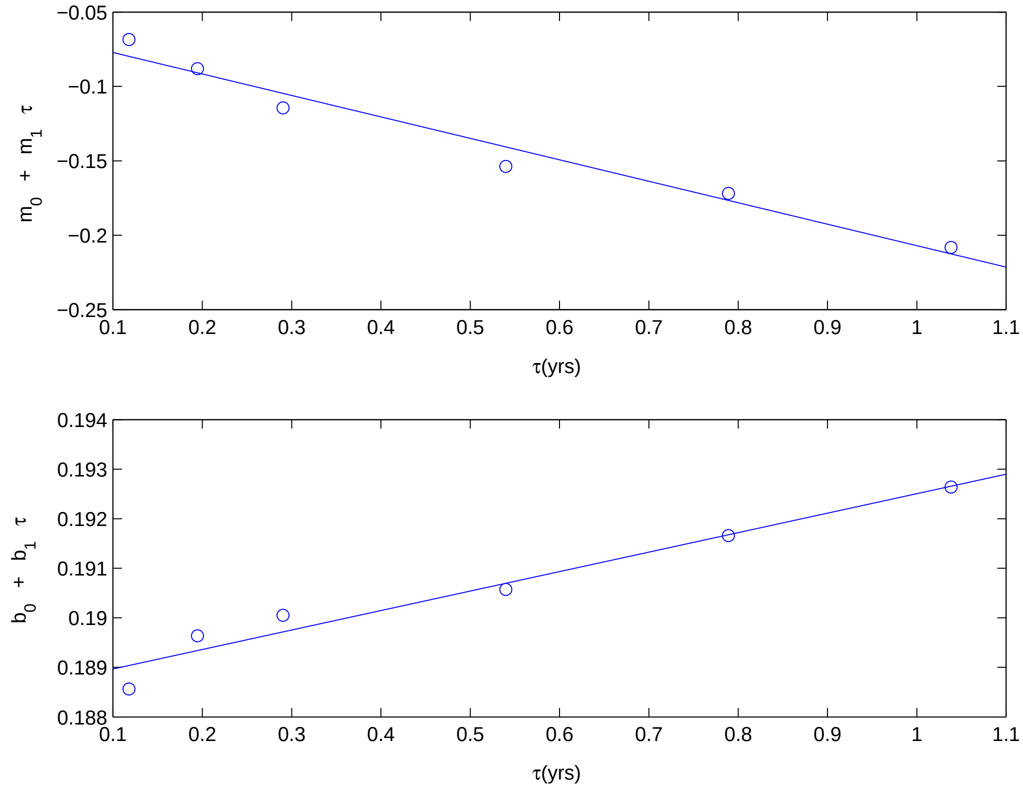
We typically fit the parameters **a**, **b**, ... by regressing in LMMR **maturity-by-maturity**, then we fit their dependence in τ.

We will see that our expansion leads to **a**, **b** which are affine in τ.

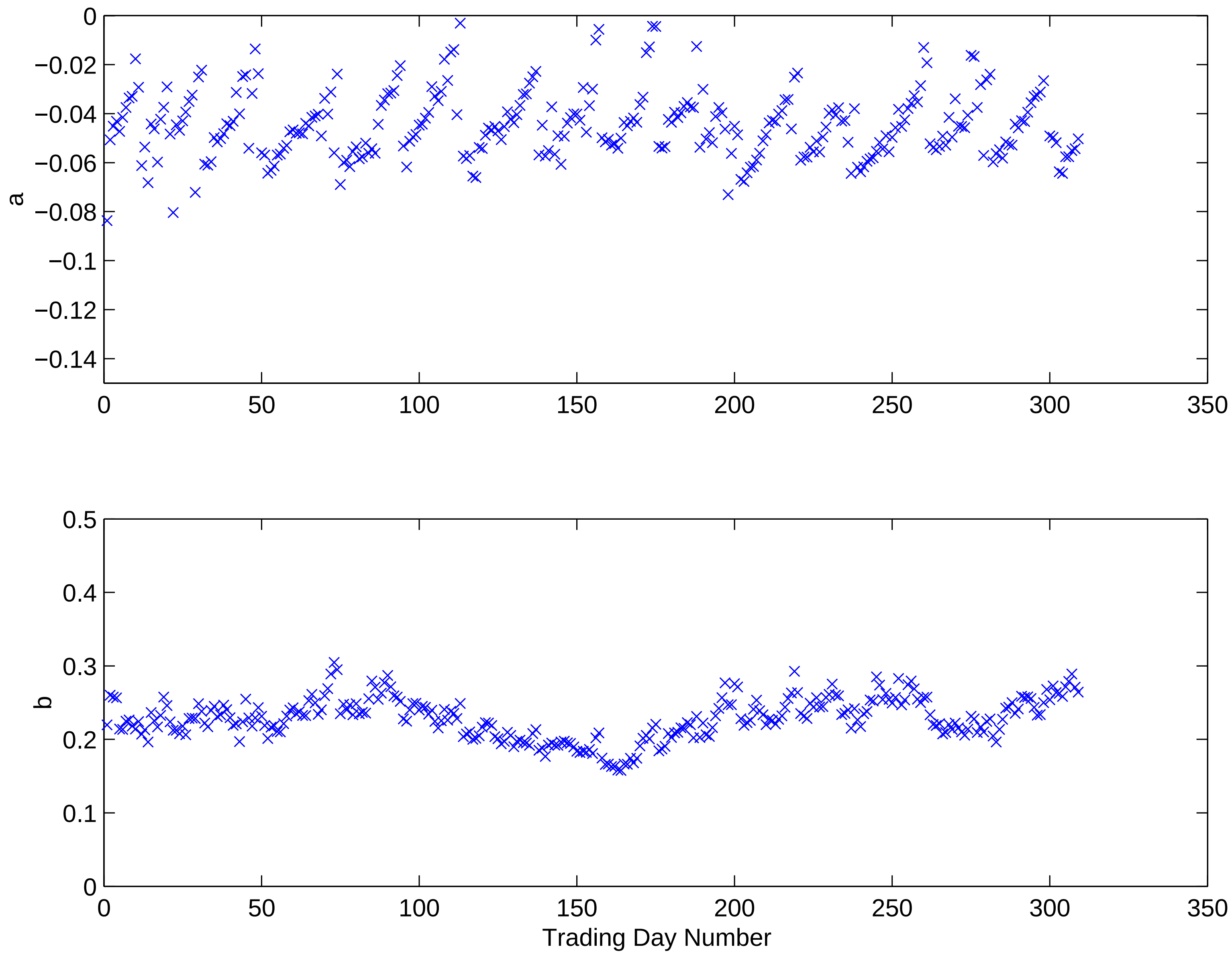
Some examples $\longrightarrow$



S&P 500 *Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation for six different maturities.*

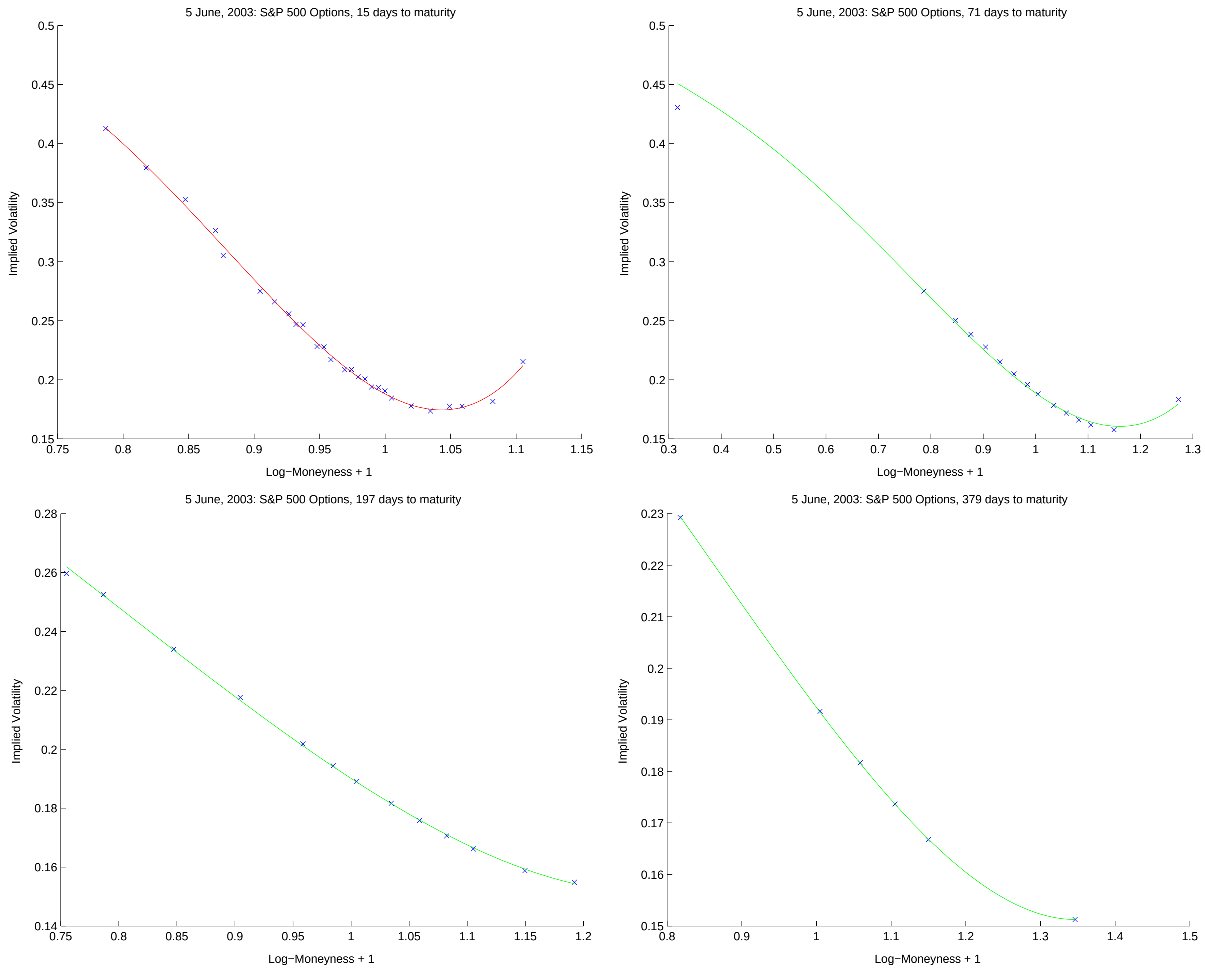


S&P 500 *Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (rep. top) figure shows the linear regression of b (resp. a) with respect to time to maturity* τ.

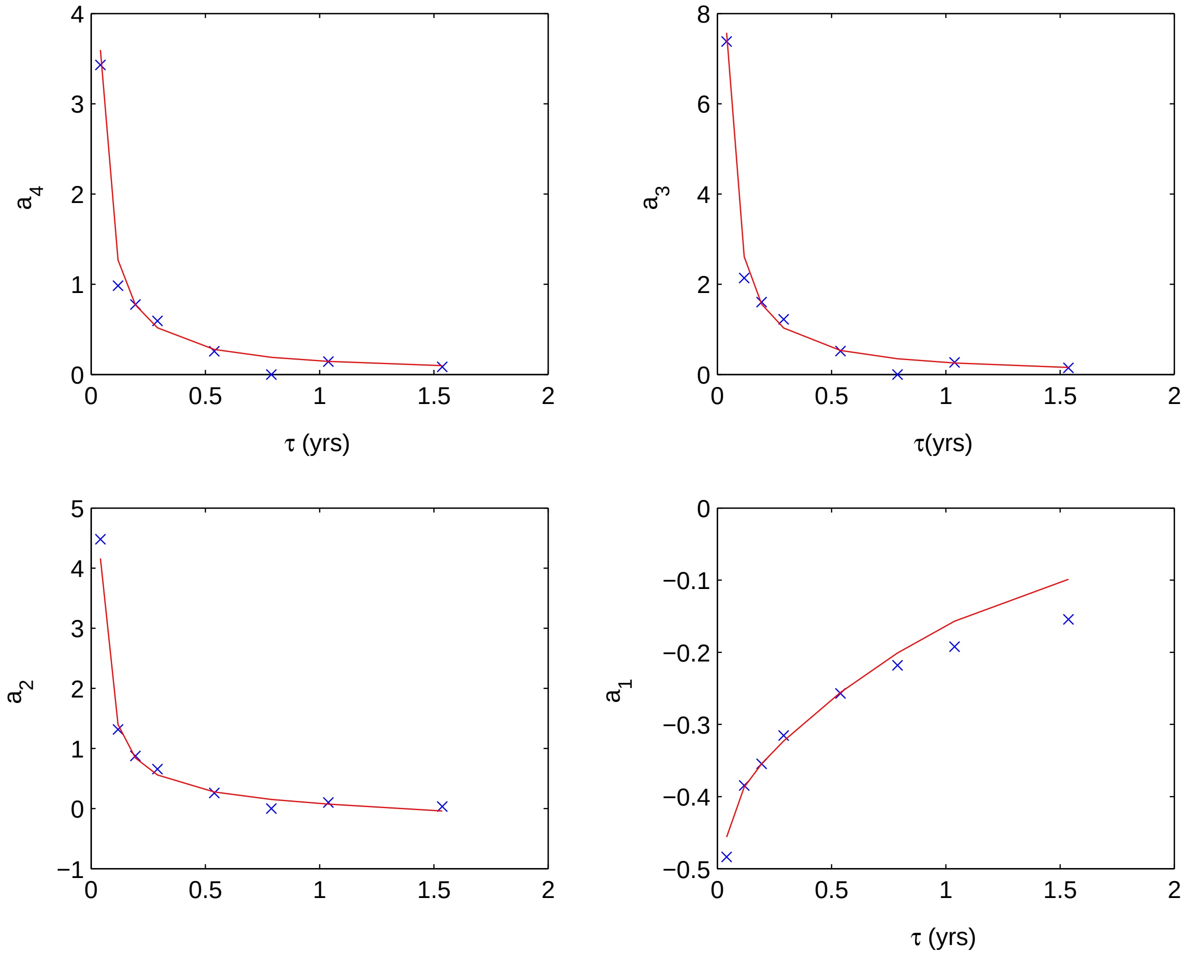

0
−0.02
−0.04
−0.06
−0.08
−0.1
−0.12
−0.14
a
0
50
100
150
200
250
300
350
0.5
0.4
0.3
0.2
0.1
0
b
Trading Day Number

Higher Order Expansion

$$\mathbf{I} \sim \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{4}} \mathbf{a_j}(\tau)\,(\mathbf{LM})^{\mathbf{j}} + \frac{\mathbf{1}}{\tau}\boldsymbol{\Phi}_{\mathbf{t}},$$



S&P 500 *Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.*



S&P 500 *Term-Structure Fit using second order approximation. Data from June 5, 2003.*

Stochastic Volatility Models

Equity for instance.

Under physical measure:

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \boldsymbol{\sigma_t} dW_t^{(0)} \\
\boldsymbol{\sigma_t} &= f(Y_t, Z_t, \cdots) \\
dY_t &= \alpha(Y_t)dt + \beta(Y_t)dW_t^{(1)} \\
dZ_t &= c(Z_t)dt + g(Z_t)dW_t^{(2)} \\
&\cdots
\end{aligned}
$$

Volatility factors can be differentiated by their **time scales**

Multiscale Stochastic Volatility Models

$$\sigma_{\mathbf{t}} = \mathbf{f}(\mathbf{Y_t}, \mathbf{Z_t})$$

- Y_t **is fast mean-reverting** (ergodic on a fast time scale):

$$dY_t = \frac{\mathbf{1}}{\varepsilon}\alpha(Y_t)dt + \frac{\mathbf{1}}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^{(1)}, \quad 0 < \varepsilon \ll 1$$

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- Z_t **is slowly varying:**

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}\, g(Z_t)dW_t^{(2)}, \quad 0 < \delta \ll 1$$

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Separation of time scales: $\varepsilon << T << 1/\delta$

(assuming f continuous in z):

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{1}{T}\int_0^T f^2(Y_t, Z_t)dt \longrightarrow \left\langle f^2(\cdot, z)\right\rangle_{\Phi_Y}$$

Local Effective Volatility: $\quad \bar{\sigma}^{\mathbf{2}}(\mathbf{z}) \equiv \left\langle \mathbf{f}^{\mathbf{2}}(\cdot, \mathbf{z})\right\rangle_{\mathbf{\Phi_Y}}$

$$\mathbf{P_0} = \mathbf{P_{BS}}(\bar{\sigma}(\mathbf{z}))$$

Market Prices of Volatility Risk

Under the **risk neutral** measure $I\!P^\star$ *chosen by the market*:

$$
\begin{aligned}
dS_t &= rS_t dt + f(Y_t, Z_t) S_t dW_t^{(0)\star} \\
dY_t &= \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda(Y_t, Z_t) \right) dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t) dW_t^{(1)\star} \\
dZ_t &= \left(\delta\ c(Z_t) - \sqrt{\delta}\ g(Z_t)\Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta}\ g(Z_t) dW_t^{(2)\star}
\end{aligned}
$$

$$
\begin{aligned}
d < W^{(0)\star}, W^{(1)\star} >_t &= \rho_1 dt \\
d < W^{(0)\star}, W^{(2)\star} >_t &= \rho_2 dt
\end{aligned}
$$

Λ and Γ: market prices of volatility risk

Pricing Equation

$$P^{\varepsilon,\delta}(t,x,y,z) \quad = \quad \mathbb{E}^{\star}\left\{e^{-r(T-t)}h(S_T)|S_t = x, Y_t = y, Z_t = z\right\}$$

Feynman–Kac:

$$\left(\frac{\mathbf{1}}{\varepsilon}\mathcal{L}_{\mathbf{Y}} + \frac{\mathbf{1}}{\sqrt{\varepsilon}}\mathcal{L}_{\rho_{\mathbf{1}},\mathbf{\Lambda}} + \mathcal{L} + \sqrt{\delta}\mathcal{L}_{\rho_{\mathbf{2}},\mathbf{\Gamma}} + \delta\mathcal{L}_{\mathbf{Z}} + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{L}_{\rho_{\mathbf{12}}}\right)P^{\varepsilon,\delta} = 0$$

$$P^{\varepsilon,\delta}(T,x,y,z) = h(x)$$

with

$$\mathcal{L} = \mathcal{L}_{BS}(f(y,z)) = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y,z)x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right)$$

Regular-Singular Perturbations

$$P^{\varepsilon,\delta} = \sum_{i,j} \varepsilon^{i/2}\, \delta^{j/2}\, P_{i,j} = P_0 + \sqrt{\varepsilon}\, P_{1,0} + \sqrt{\delta}\, P_{0,1} + \cdots$$

$$\mathcal{L}_{BS}(\bar{\sigma}(z))P_0 = 0, \quad P_0(T,x) = h(x) \quad \Longrightarrow \mathbf{P_0} = \mathbf{P_{BS}}(\bar{\sigma}(\mathbf{z}))$$

P_0 is independent of y and z is a parameter.

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$$\begin{aligned}
\mathcal{L}_{\mathbf{BS}}(\bar\sigma(\mathbf{z}))\left(\sqrt{\varepsilon}\mathbf{P_{1,0}}\right) + \mathbf{V_2^\varepsilon D_2 P_{BS}} + \mathbf{V_3^\varepsilon D_1 D_2 P_{BS}} &= 0 \\
\mathcal{L}_{\mathbf{BS}}(\bar\sigma(\mathbf{z}))\left(\sqrt{\delta}\mathbf{P_{0,1}}\right) + \mathbf{2}\left(\mathbf{V_0^\delta}\partial_\sigma \mathbf{P_{BS}} + \mathbf{V_1^\delta D_1}\partial_\sigma \mathbf{P_{BS}}\right) &= 0 \\
\mathbf{P_{1,0}(T,x)} = \mathbf{P_{0,1}(T,x)} &= 0
\end{aligned}$$

$\mathbf{V_0^\delta}$ and $\mathbf{V_2^\varepsilon}$ are volatility level adjustments due to $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ resp.
$\mathbf{V_1^\delta}$ and $\mathbf{V_3^\varepsilon}$ are skew parameters proportional to $\rho_\mathbf{2}$ and $\rho_\mathbf{1}$ resp.

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Important: these Black-Scholes equations will hold for exotic options with **additional boundary conditions**, but with the **same group parameters V's**

Explicit formulas for Vanilla European Options

Notation: $T - t = \tau$

$$\sqrt{\varepsilon}\mathbf{P_{1,0}} = \tau \left(\mathbf{V_2^{\varepsilon} D_2 P_{BS}} + \mathbf{V_3^{\varepsilon} D_1 D_2 P_{BS}} \right)$$

easily checked by using $\mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS}$

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$$\sqrt{\delta}\mathbf{P_{0,1}} = \tau\left(\mathbf{V_0^\delta}\partial_\sigma\mathbf{P_{BS}} + \mathbf{V_1^\delta D_1}\partial_\sigma\mathbf{P_{BS}}\right)$$

easily checked by using first $\partial P_{BS} = \tau\bar{\sigma} D_2 P_{BS}$ and then $\mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS}$

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easily checked by using $\partial P_{BS} = \tau\bar{\sigma} D_2 P_{BS}$ and then $\mathcal{L}_{BS} D_i = D_i \mathcal{L}_{BS}$.

- **Back to our expansion** $\longrightarrow$

$$P = P_0 + v_0 \partial_\sigma P_0 + v_1 D_1 \partial_\sigma P_0 + v_2 D_2 P_0 + v_3 D_1 D_2 P_0 + \cdots$$

$$\mathbf{v_0} = \tau \mathbf{V_0^{\delta}}\,, \quad \mathbf{v_1} = \tau \mathbf{V_1^{\delta}}$$

$$\mathbf{v_2} = \tau \mathbf{V_2^{\varepsilon}}\,, \quad \mathbf{v_3} = \tau \mathbf{V_3^{\varepsilon}}$$

In terms of calibration to implied volatilities $\longrightarrow$

Implied Volatility Calibration Formulas

$$\underbrace{\bar{\sigma} + \frac{V_2}{\bar{\sigma}} + \frac{V_3}{2\bar{\sigma}}(1 - \frac{2r}{\bar{\sigma}^2}) + \tau\left(V_0 + \frac{V_1}{2}(1 - \frac{2r}{\bar{\sigma}^2})\right)}_{\text{intercept } b} \quad + \quad \underbrace{\left(\frac{V_3}{\bar{\sigma}^3} + \tau\frac{V_1}{\bar{\sigma}^2}\right)}_{\text{slope } a} \quad \text{LMMR}$$

Either

- one estimates $\bar{\sigma}$ **from historical data** (preferred for **hedging** where V_0 and V_2 do not appear), and then fitting maturity-by-maturity and regressing in τ, one gets:
  1. V_1 and V_3 from the slope a
  2. V_0 and V_2 from the intercept b

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  1. V_1 and V_3 from the slope a
  2. V_0 and V_2 from the intercept b
- or one uses the **adjusted effective volatility** $\sigma^\star \equiv \sqrt{\bar{\sigma}^2 + 2V_2}$ calibrated from **option data** , along with V_0, V_1, and V_3 (preferred for **pricing**):

$$\sigma^\star + \frac{V_3}{2\sigma^\star}(1 - \frac{2r}{\bar{\sigma}^{\star 2}}) + \tau\left(V_0 + \frac{V_1}{2}(1 - \frac{2r}{\sigma^{\star 2}})\right) + \left(\frac{V_3}{\sigma^{\star 3}} + \tau\frac{V_1}{\sigma^{\star 2}}\right)\text{LMMR}$$

Back to the Wish List: Accuracy

If the payoff function h is smooth:

$$
\begin{aligned}
& P^{\varepsilon,\delta} \\
&= \left(P_0 + \sqrt{\varepsilon}P_{1,0} + \varepsilon P_{2,0} + \varepsilon^{3/2}P_{3,0}\right) + \sqrt{\delta}\left(P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1}\right) + R^{\varepsilon,\delta} \\
&= \left(\mathbf{P_0} + \sqrt{\varepsilon}\mathbf{P_{1,0}} + \sqrt{\delta}\mathbf{P_{0,1}}\right) + \mathcal{O}(\varepsilon+\delta) + R^{\varepsilon,\delta}
\end{aligned}
$$

then the **residual** $R^{\varepsilon,\delta}$ satisfies

$$
\begin{aligned}
\mathcal{L}^{\varepsilon,\delta}R^{\varepsilon,\delta} &= \mathcal{O}(\varepsilon+\delta) \\
R^{\varepsilon,\delta}(T) &= \mathcal{O}(\varepsilon+\delta)
\end{aligned}
$$

and therefore $R^{\varepsilon,\delta} = \mathcal{O}(\varepsilon+\delta)$.

If h is **non-smooth** (call option in particular), then use a careful **regularization**.

Path-Dependent Derivatives (Barrier, Asian,...)

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the **implied volatility surface**
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow \mathbf{P_0} = \mathbf{P_{BS}}(\sigma^\star)$$

- Use V_0, V_1 and V_3 to compute the **source**

$$\mathbf{2}\left(\mathbf{V_0}\partial_\sigma \mathbf{P^\star_{BS}} + \mathbf{V_1 D_1}\partial_\sigma \mathbf{P^\star_{BS}}\right) + \mathbf{V_3 D_1 D_2 P^\star_{BS}}$$

- Get the **correction** by solving the SAME PROBLEM with **zero boundary conditions** and the **source**.

American Options

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the **implied volatility surface**
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow \mathbf{P}^\star \textbf{ and the free boundary } \mathbf{x}^\star(\mathbf{t})$$

- Use V_0, V_1 and V_3 to compute the **source**

$$\mathbf{2}\left(\mathbf{V_0}\partial_\sigma \mathbf{P}^\star_{\mathbf{BS}} + \mathbf{V_1}\mathbf{D_1}\partial_\sigma \mathbf{P}^\star_{\mathbf{BS}}\right) + \mathbf{V_3}\mathbf{D_1}\mathbf{D_2}\mathbf{P}^\star_{\mathbf{BS}}$$

- Get the **correction** by solving the corresponding problem with **fixed boundary $\mathbf{x}^\star(\mathbf{t})$**, **zero boundary conditions** and the **source**.

Cost of the Black-Scholes Hedging Strategy

$$
\begin{aligned}
P_{BS}(T, S_T) &= h(S_T) \\
P_{BS}(t, S_t) &= a_t S_t + b_t e^{rt}, \quad a_t = \partial_x P_{BS}
\end{aligned}
$$

Infinitesimal cost:

$$
dP_{BS}(t, S_t) - \underbrace{\left(a_t dS_t + r b_t e^{rt} dt\right)}_{\text{self-financing part}} = \frac{1}{2}\left(f^2(Y_t, Z_t) - \sigma^2\right) D_2 P_{BS}(t, S_t) dt
$$

Cumulative financing cost:

$$
\mathbf{E_{BS}(t) = \frac{1}{2} \int_0^t e^{-rs} \left(f^2(Y_s, Z_s) - \sigma^2\right) D_2 P_{BS}(s, S_s) ds}
$$

Choice of σ ?

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Since $\mathbf{Y_t}$ is **fast mean-reverting** ($\varepsilon << 1$), integrals like

$$\int_{\mathbf{0}}^{\mathbf{t}} \left(\mathbf{f^2}(\mathbf{Y_s}, \mathbf{Z_s}) - \sigma^{\mathbf{2}}\right) \mathbf{\Psi_s ds} \quad \text{will be small with } \varepsilon \text{ if}$$

$$\sigma^{\mathbf{2}} = \bar{\sigma}^{\mathbf{2}}(\mathbf{z}) = \langle \mathbf{f^2}(\cdot, \mathbf{z}) \rangle_{\mathbf{\Phi(Y)}}$$

Therefore **two choices**:

- $\sigma^2 = \bar{\sigma}^2(Z_t)$ and $P_{BS} = P_{BS}(t, S_t; \bar{\sigma}(Z_t))$, in which case $\bar{\sigma}(Z_t)$ needs to be estimated continuously (and dP_{BS} revisited)
- $\sigma^2 = \bar{\sigma}^2(Z_0)$ and $P_{BS} = P_{BS}(t, S_t; \bar{\sigma}(Z_0))$ with

$$\mathbf{f^2}(\mathbf{Y_s}, \mathbf{Z_s}) - \sigma^{\mathbf{2}} = \left(\mathbf{f^2}(\mathbf{Y_s}, \mathbf{Z_s}) - \bar{\sigma}^{\mathbf{2}}(\mathbf{Z_t})\right) + \left(\bar{\sigma}^{\mathbf{2}}(\mathbf{Z_t}) - \bar{\sigma}^{\mathbf{2}}(\mathbf{Z_0})\right)$$

  in which case parameters are frozen at time zero, an additional cost of order $\sqrt{\delta}$ comes from the second term (offset in practice by re-calibration at $\sqrt{\delta}$-frequency).

Corrected Hedging Strategy

A careful analysis of the cost shows

$$\begin{aligned} \mathbf{E_0(t)} &= \frac{1}{2}\int_0^t \mathbf{e^{-rs}}\left(\mathbf{f^2}(\mathbf{Y_s},\mathbf{Z_s}) - \bar{\sigma}^2(\mathbf{Z_t})\right)\mathbf{D_2P_{BS}}(\mathbf{s},\mathbf{S_s})\mathbf{ds} \\ &= \sqrt{\varepsilon}\left(\mathbf{B_t^\varepsilon} + \mathbf{M_t^\varepsilon}\right) + \mathcal{O}(\varepsilon + \delta), \end{aligned}$$

where $\mathbf{M_t^\varepsilon}$ is a martingale, and

$$\mathbf{B_t^\varepsilon} = -\frac{\rho_1}{2}\int_0^t \mathbf{e^{-rs}}\beta(\mathbf{Y_s})\frac{\partial\phi}{\partial \mathbf{y}}\mathbf{f}(\mathbf{Y_s},\mathbf{Z_s})\mathbf{D_1D_2P_{BS}}(\mathbf{s},\mathbf{S_s})\mathbf{ds}$$

is a bounded variation **bias** which can be **compensated** by using the **corrected hedging ratio $\mathbf{a_t}$** given by

$$\partial_\mathbf{x}\mathbf{P_{BS}} + (\mathbf{T}-\mathbf{t})\mathbf{V_3}\partial_\mathbf{x}\mathbf{D_1D_2P_{BS}} + (T-t)V_1\partial_x D_1 \partial_\sigma P_{BS}$$

The last term compensates for the bias generated by $\bar{\sigma}^2(Z_t) - \bar{\sigma}^2(Z_0)$

Examples of other:

- **Models**
- **Regimes**
- **Applications**

A Model with Volatility Time-Scale of Order One

In the model $\sigma_\mathbf{t} = \mathbf{f}(\mathbf{Y_t}, \mathbf{Z_t})$, if one wants to:

- keep $\mathbf{Y}$ fast mean-reverting
- let $\mathbf{Z}$ be on a time scale comparable to maturity (or add one such factor)
- keep the computational tractability

then, one needs to make sure that the SV model $\bar{\sigma}^\mathbf{2}(\mathbf{Z_t})$ is tractable.

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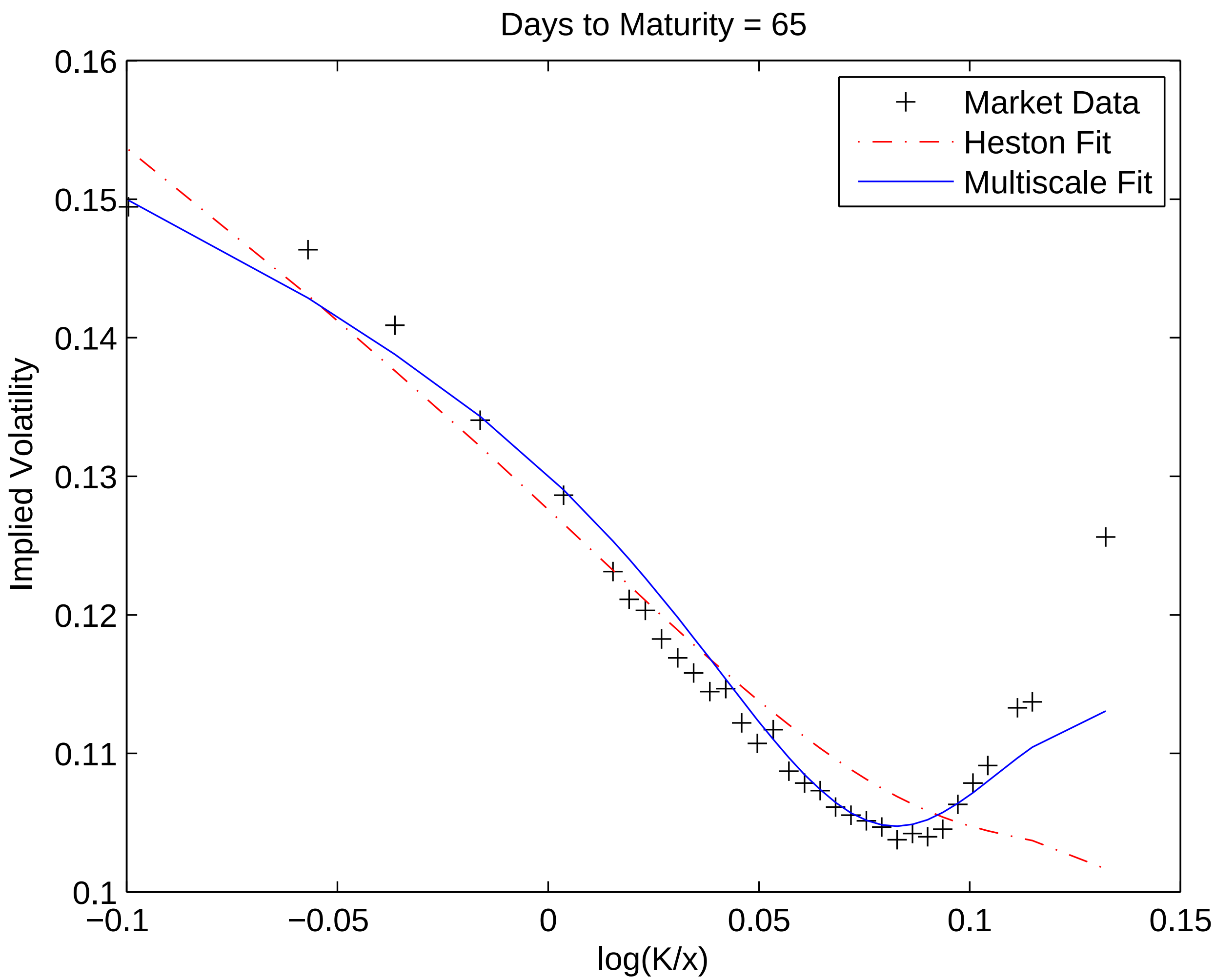
An interesting choice is the **Heston model**:
"A Fast Mean-Reverting Correction to Heston Stochastic Volatility Model" with **Matthew Lorig** (PhD student, UCSB), where we develop this idea.

An example of fit $\longrightarrow$



SPX Implied Volatilities from May 17, 2006

Fast Mean-Reverting SV and Short Maturities

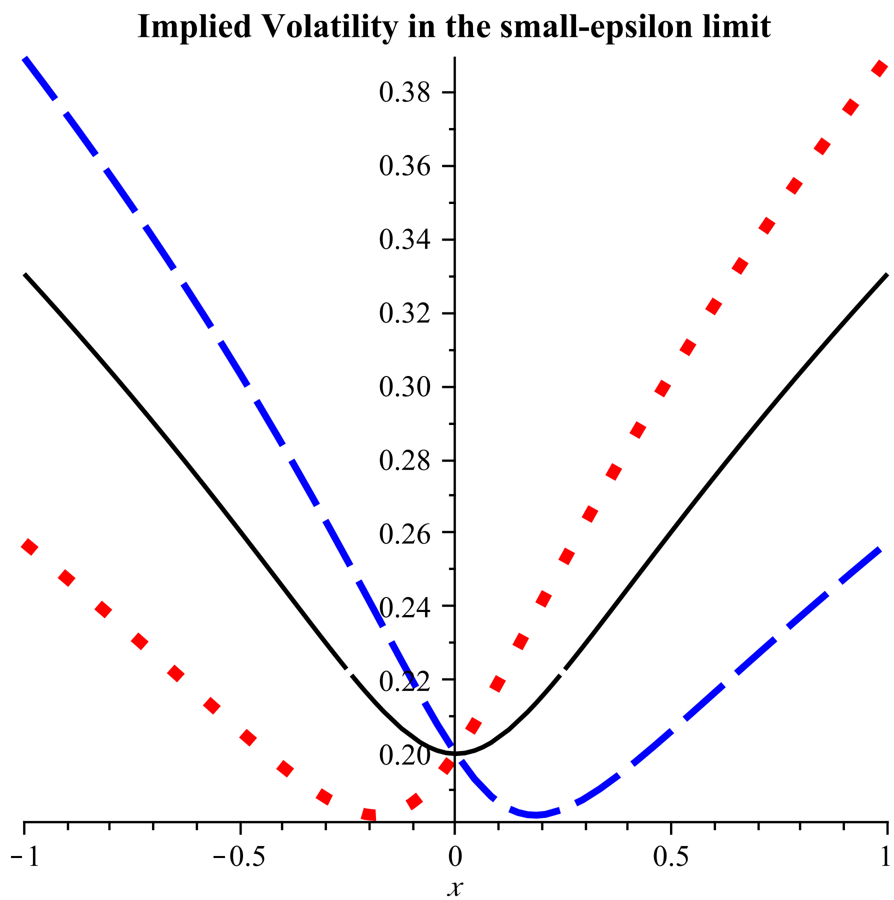
If the time scale of the fast mean-reverting factor **Y** is $\varepsilon << \mathbf{1}$, and if the maturity of interest is small but still large compared with ε, then, one can consider the **regime**

$$\varepsilon << \mathbf{T} \sim \sqrt{\varepsilon} << \mathbf{1}$$

It involves a non-trivial mixture of **Large Deviation** (short maturity) and **Homogenization** (fast mean reverting coefficient):

“Short maturity asymptotics for a fast mean reverting Heston stochastic volatility model” with **Jin Feng and Martin Forde** (SIAM Journal on Financial Mathematics, Vol. 1, 2010).

Interestingly, in this regime and for this model, we derive explicit formulas for the limiting implied volatility which looks like $\longrightarrow$



Three parameters which control the implied volatility skew's *level* (θ), *slope* (ρ) and *convexity* (ν/κ).

A Cool Application to Forward-Looking Betas

Discrete time CAPM model:

$$R_a - R_f = \beta_{\mathbf{a}}(R_M - R_f) + \epsilon_a$$

Christoffersen, Jacobs, and Vainberg (2008, McGill University):

$$\beta_a = \left(\frac{SKEW_a}{SKEW_M}\right)^{\frac{1}{3}} \left(\frac{VAR_a}{VAR_M}\right)^{\frac{1}{2}}$$

where VAR and $SKEW$ are **variance** and **risk-neutral skewness**

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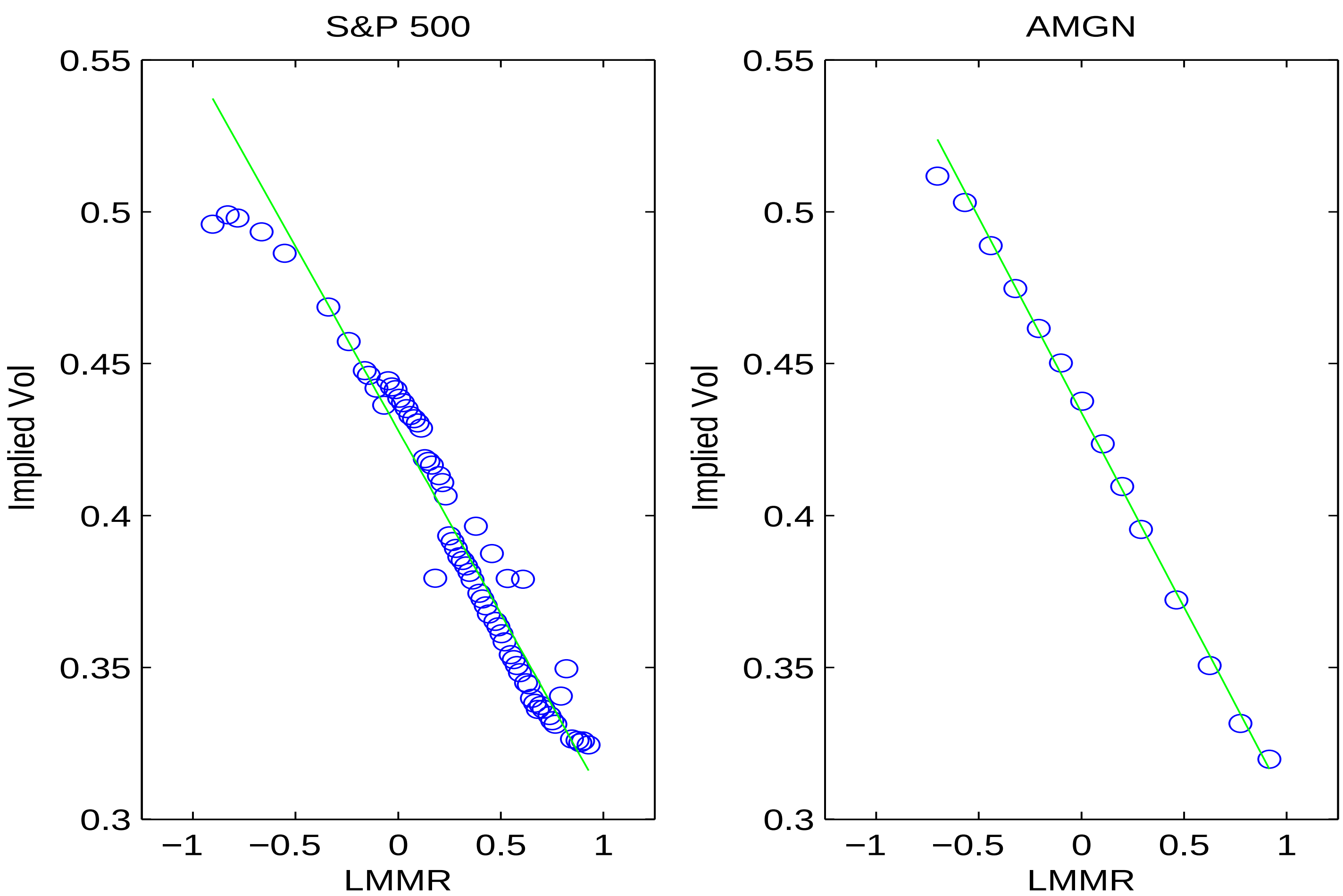
With **Eli Kollman** (PhD 2009, UCSB), we propose in

"Calibration of Stock Betas from Skews of Implied Volatilities"

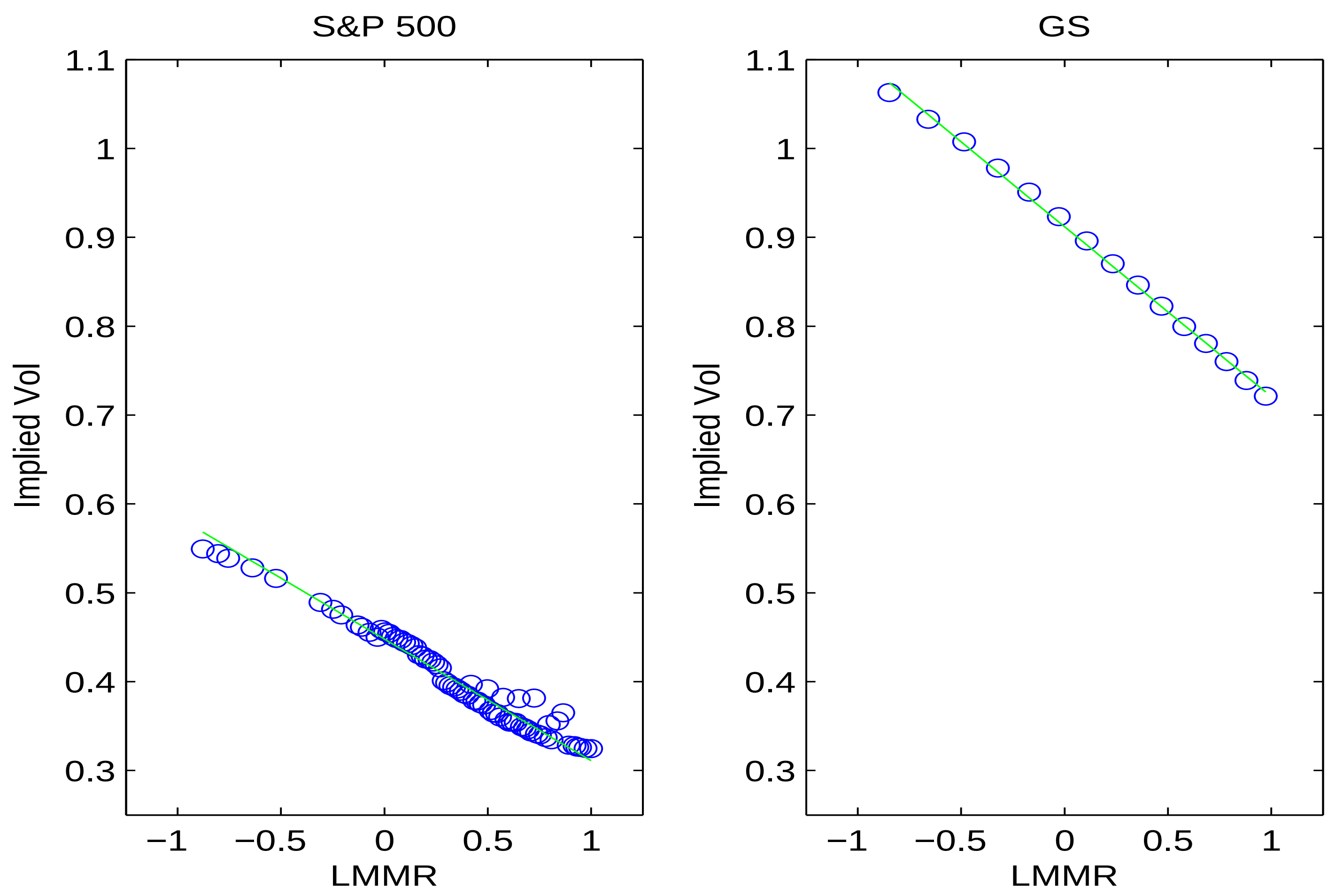
(Applied Mathematical Finance, 2010):

$$\hat{\beta}_{\mathbf{a}} = \left(\frac{\mathbf{V_3^{a,\epsilon}}}{\mathbf{V_3^{M,\epsilon}}}\right)^{\mathbf{1/3}} = \left(\frac{\mathbf{a^{a,\epsilon}}}{\mathbf{a^{M,\epsilon}}}\right)^{\mathbf{1/3}} \left(\frac{\mathbf{b^{a*}}}{\mathbf{b^{M*}}}\right)$$

LMMR fits (2/18/2009): S&P500 and Amgen, beta estimate is 1.03



LMMR fits (2/19/2009): S&P500 and Goldman Sachs, beta estimate is 2.28



THANKS FOR YOUR ATTENTION