Better than Dynamic Mean-Variance Policy in Market with ALL Risky Assets

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Outline

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Dynamic Mean-Variance Portfolio Selection

- In the early 1950s, Markowitz published his pioneering work on single-period mean-variance portfolio selection, which has paved a foundation of modern financial analysis.
- [Li and Ng 2000] solved the mean-variance formulation of the multi-period portfolio selection problem by adopting an embedding scheme. In the same year, [Zhou and Li 2000] also solved the mean-variance formulation in continuous-time by adopting the same embedding scheme.
- [Zhu, Li and Wang 2003] investigated the wealth reduction phenomena associated with the optimal multi-period mean-variance policy. [Basak and Chabakauri 2008] also recognized that investors may have incentives to deviate from the optimal dynamic mean-variance policy, which is termed pre-committed optimal policy, before reaching the terminal time.

Time consistent dynamic risk measure

- [Artzner, Delbaen, Eber and Heath 1997, 1999] introduced coherent risk measures. [Föllmer and Schied 2002], [Frittelli and Rosazza Gianin 2002] further introduced convex risk measure.
- Although "Time Consistency" requirements for dynamic risk measure introduced by [Rosazza Gianin 2002], [Boda and Filar 2006], [Artzner, Delbaen, Eber, Heath, Ku 2007], [Jobert and Rogers 2008] read differently, they all have their essence rooted in Bellman's dynamic programming.
- [Cui, Li, Wang and Zhu 2009] introduced the concept of Time Consistency in Efficiency for mean-risk model, which is rooted in multi-objective dynamic programming, and derived a better revised mean-variance policy in markets with a riskless asset.

Time consistency in efficiency

Definition (Time Consistency in Efficiency)

Assume that $(\pi_0^*, \cdots, \pi_{T-1}^*)$ *is the optimal policy of*

$$\min_{\pi_0, \cdots, \pi_{T-1}} \{\mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0) + \lambda E(x_T \mid \pi_0, \ldots, \pi_{T-1}, x_0)\}, \ \ \lambda \leq 0.$$

Risk measure $\mathcal{M}$ *(and its overall optimal policy) is said to satisfy* ***time consistency in efficiency****, if for all* $t = 1, \ldots, T-1$*,*

$$(\pi_t^*, \ldots, \pi_{T-1}^*)$$
$$\in \arg\min_{\pi_t, \ldots, \pi_{T-1}} \{\mathcal{M}_{t-T}(\pi_t, \ldots, \pi_{T-1} \mid x_t) + \lambda_t E(x_T \mid \pi_t, \ldots, \pi_{T-1}, x_t)\},$$

holds for some nonpositive λ_t *and any possible wealth level* x_t*.*

Outline

Market Setting

- Consider a capital market consisted of only $n+1$ risky assets within a finite time horizon T.
- $\mathbf{e_t} = (e_t^0, ..., e_t^n)'$: the vector of random total return rates of the $n+1$ risky assets during period t with known first two moments, the mean and the covariance.
- Vectors $\mathbf{e_t}$, t = 0, 1, . . ., T - 1, are assumed to be statistically independent.
- x_0: a given initial wealth level, .
- x_t: the wealth level at the beginning of the t-th time period.
- u_t^i (i = 1, 2, . . ., n): the amount invested in the ith risky asset at the beginning of the t-th time period.

Problem Formulation

The dynamic mean-variance portfolio problem is given by

$$
\begin{aligned}
(MV) \quad & \min \quad Var(x_T|x_0) + \lambda E(x_T|x_0) \\
& \text{s.t.} \quad x_{t+1} = e_t^0 x_t + \mathbf{P_t'u_t}, \quad t = 0, 1, ..., T-1, \qquad (1) \\
& \qquad\ \ x_0 > 0 \text{ is given,}
\end{aligned}
$$

where

$$\mathbf{P_t} = (P_t^1, P_t^2, ..., P_t^n)' = ((e_t^1 - e_t^0), (e_t^2 - e_t^0), ..., (e_t^n - e_t^0))'$$

satisfies

$$
\begin{aligned}
E(\mathbf{P_t P_t'}) \succ 0, \quad & \forall t = 0, 1, ..., T-1, \\
E((e_t^0)^2) - E(e_t^0 \mathbf{P_t'}) E^{-1}(\mathbf{P_t P_t'}) E(e_t^0 \mathbf{P_t}) > 0, \quad & \forall t = 0, 1, ..., T-1.
\end{aligned}
$$

Pre-committed Optimal Policy

The pre-committed optimal policy for (MV) [Li and Ng 2000]:

$$\mathbf{u}_\mathbf{t}^*(x_t) = -E^{-1}(\mathbf{P_t P_t'})E(e_t^0\mathbf{P_t})x_t + \Gamma\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t}).$$

where $\Gamma = \frac{1}{2}\left(b_0 x_0 - \frac{\nu_0 \lambda}{2a_0}\right)$ is termed risk attitude parameter. Furthermore, [Li and Ng 2000] give the minimum variance set of (MV) explicitly as follows,

$$Var(x_T|x_0) = \frac{a_0}{\nu_0^2}(E(x_T|x_0) - (\mu_0 + b_0\nu_0)x_0)^2 + c_0 x_0^2. \qquad (2)$$

It is easy to verify that, when $E(x_T|x_0) \geq (\mu_0 + b_0\nu_0)x_0$, the mean-variance pair is efficient.

Parameters

Define the following parameters:

$$B_t = E(\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(\mathbf{P_t}) > 0,$$
$$A_t^1 = E(e_t^0) - E(\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(e_t^0\mathbf{P_t}),$$
$$A_t^2 = E((e_t^0)^2) - E(e_t^0\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(e_t^0\mathbf{P_t}) > 0,$$
$$\mu_t = \prod_{k=t}^{T-1} A_t^1, \quad \nu_t = \sum_{k=t}^{T-1}\left(\prod_{j=k+1}^{T-1} A_j^1\right) B_k^1, \quad \tau_t = \prod_{k=t}^{T-1} A_k^2,$$
$$a_t = \frac{\nu_t}{2} - (\nu_t)^2, \quad b_t = \frac{\mu_t\nu_t}{a_t} = \frac{2\mu_t}{1-2\nu_t}, \quad c_t = \tau_t - (\mu_t)^2 - a_t(b_t)^2.$$

Outline

Three dimensional objective space

The pre-committed mean-variance efficient pair, which satisfies equation (2) for $(E(x_T|x_0), Var(x_T|x_0))$, is Pareto-optimal in the objective space of

$$\{\max\,(\text{expected terminal wealth}),\ \min\,(\text{variance of the terminal wealth})\}.$$

In the real world, we'd better consider the efficiency in an expanded three-dimensional objective space:

$$\{\min\,(\text{initial investment level}),\ \max\,(\text{expected terminal wealth}),\ \min\,(\text{variance of the terminal wealth})\}.$$

Pseudo Efficiency (Type 1)

Consider the revised mean-variance portfolio selection,

$$\begin{aligned}(RMV_1) \qquad \min \quad & Var(x_T|y_0) + \lambda E(x_T|y_0) \\ \text{s.t.} \quad & x_{t+1} = e_t^0 x_t + \mathbf{P_t'u_t}, \quad t = 1, 2, \ldots, T-1, \\ & x_1 = e_0^0 y_0 + \mathbf{P_0'u_0}, \\ & y_0 \le x_0. \end{aligned}$$

Definition

For a wealth level x_0, if an efficient mean-variance pair for (MV) is dominated by a mean-variance pair of problem (RMV_1), i.e.,

$$(-x_0, E(x_T|x_0), -Var(x_T|x_0)) \prec (-y_0, E(x_T|y_0), -Var(x_T|y_0)), \quad (3)$$

the given T-period mean-variance pair is termed pseudo efficient (type 1).

The Existence of Pseudo Efficiency (Type 1)

Proposition

Pseudo efficiency condition (3) $\iff x_0 > \bar{x}_0^* = \Gamma\mu_0/\tau_0.$
For a given positive initial wealth x_0*, condition* $x_0 > \bar{x}_0^*$ *does not hold when*

$$\lambda = \begin{cases} \leq \dfrac{2(\mu_0^2 - (1-2\nu_0))}{\mu_0} x_0 < 0, & \text{if } \mu_0 > 0, \\ \geq \dfrac{2(\mu_0^2 - (1-2\nu_0))}{\mu_0} x_0 > 0, & \text{if } \mu_0 < 0. \end{cases}$$

Remark

The concept of pseudo efficiency (type 1) can be extended to truncated $(T-s)$*-period problem.*

The First Type of Revised Policies

Proposed T-period revised portfolio policy for (MV):

$$\widehat{\mathbf{u}}_{\mathbf{k}}^{*}(\widehat{x}_k) = -E^{-1}(\mathbf{P_k}\mathbf{P_k'})E(e_k^0\mathbf{P_k})\widehat{x}_k + \Gamma_k\left(\frac{\mu_{k+1}}{\tau_{k+1}}\right)E^{-1}(\mathbf{P_k}\mathbf{P_k'})E(\mathbf{P_k}); \qquad (4)$$

$$\widehat{x}_k = \begin{cases} \bar{x}_k, & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ -\bar{x}_k + \dfrac{2\mu_k(\mu_k\bar{x}_k + 2\nu_k\Gamma_{k-1})}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases}$$

$$\bar{x}_0 = x_0$$

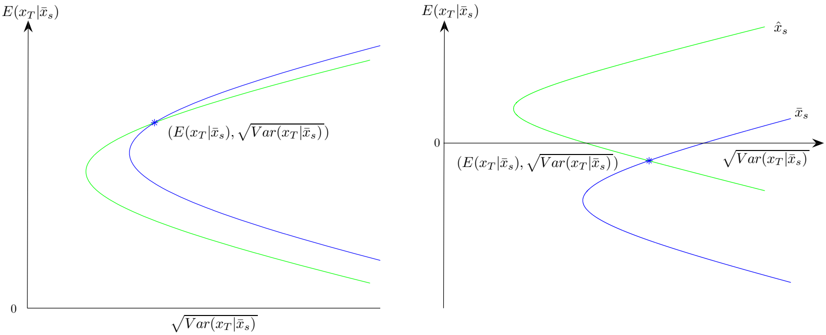
$$\bar{x}_{k+1} = e_k^0\widehat{x}_k + \mathbf{P}_k'\widehat{\mathbf{u}}_{\mathbf{k}}^{*}(\widehat{x}_k),$$

$$\Gamma_k = \begin{cases} \Gamma_{k-1}, & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ \Gamma_{k-1} + \dfrac{2\mu_k\tau_k(\bar{x}_k - \bar{x}_k^*)}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases}$$

$$\Gamma_{-1} = \frac{1}{2}\left(b_0x_0 - \frac{\lambda_0\nu_0}{2a_0}\right)$$

$$\bar{x}_k^* = \frac{\Gamma_{k-1}\mu_k}{\tau_k}.$$

Scheme Illustration of the First Revised Policy



(a) $\mu_k > 0$ (b) $\mu_k < 0$

Figure: The scheme of the first revised policy

Performance

- The first type of revised policies keeps the conditional mean and variance unchanged, thus achieving the same mean-variance pair as does the pre-committed optimal mean-variance policy of the T-period problem (MV), while having a possibility to take positive free cash flow stream, $\{\bar{x}_k - \widehat{x}_k\}$, out of the market during the investment process, i.e.,

$$E(\bar{x}_T|x_0)|_{\hat{\mathbf{u}}^*} = E(x_T|x_0)|_{\mathbf{u}^*},$$
$$Var(\bar{x}_T|x_0)|_{\hat{\mathbf{u}}^*} = Var(x_T|x_0)|_{\mathbf{u}^*},$$
$$P\{\cup_{k=1}^{N-1}[(\bar{x}_k - \widehat{x}_k)^+ > 0] \mid x_0\} > 0.$$

Pseudo Efficiency (Type 2)

Consider the revised mean-variance portfolio selection,

$$
\begin{array}{lll}
(RMV_2) & \min & Var(x_T + x_0 - y_0 | x_0) + E(x_T + x_0 - y_0 | x_0) \\
 & \text{s.t.} & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 1, 2, \ldots, T-1, \\
 & & x_1 = e_0^0 y_0 + \mathbf{P}_0' \mathbf{u}_0, \\
 & & y_0 \leq x_0.
\end{array}
$$

Definition

For a wealth level x_0, if an efficient mean-variance pair for (MV) is not pseudo efficient (type 1) and is, however, dominated by a total mean-variance pair of problem (RMV_2), i.e.,

$$(E(x_T|x_0), -Var(x_T|x_0)) \prec (E(x_T + x_0 - y_0|x_0), -Var(x_T + x_0 - y_0|x_0)),$$

then the given T-period mean-variance pair is called pseudo efficient (type 2).

The Existence of Pseudo Efficiency (Type 2)

Proposition

Pseudo efficiency (type 2) condition

$$\iff \quad (\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\Gamma.$$

For a given positive initial wealth x_0*, condition* $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\Gamma$ *does not hold when*

$$\lambda = \begin{cases} \leq \dfrac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 < 0, & \text{if } \mu_0 > 1 - 2\nu_0, \\ \geq \dfrac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 > 0, & \text{if } \mu_0 < 1 - 2\nu_0. \end{cases}$$

The Second Type of Revised Policies

Proposed T-period revised portfolio policy, $\check{\mathbf{u}}_{\mathbf{k}}^{*}(\check{x}_k)$, $k = 0, \ldots, T-1$:

$$\check{\mathbf{u}}_{\mathbf{k}}^{*}(\check{x}_k) = -E^{-1}(\mathbf{P_k P_k'})E(e_k^0 \mathbf{P_k})\check{x}_k + \overline{\Gamma}_k \left(\frac{\mu_{k+1}}{\tau_{k+1}}\right) E^{-1}(\mathbf{P_k P_k'})E(\mathbf{P_k}); \quad (5)$$

$$\check{x}_k = \begin{cases} \widetilde{x}_k, & \text{if } (\tau_k - \mu_k)\widetilde{x}_k \le (\mu_k - 1 + 2\nu_k)\overline{\Gamma}_{k-1}, \\ \dfrac{(\mu_k - 1 + 2\nu_k)[(\mu_k - 1)\widetilde{x}_k + 2\nu_k \overline{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\widetilde{x}_k > (\mu_k - 1 + 2\nu_k)\overline{\Gamma}_{k-1}, \end{cases}$$

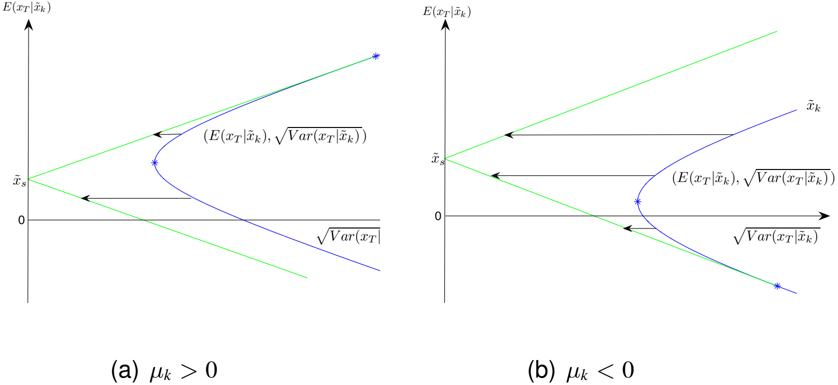
$$\widetilde{x}_0 = x_0$$

$$\widetilde{x}_{k+1} = e_k^0 \check{x}_k + \mathbf{P}_k' \check{\mathbf{u}}_{\mathbf{k}}^{*}(\check{x}_k),$$

$$\overline{\Gamma}_k = \begin{cases} \overline{\Gamma}_{k-1}, & \text{if } (\tau_k - \mu_k)\widetilde{x}_k \le (\mu_k - 1 + 2\nu_k)\overline{\Gamma}_{k-1}, \\ \dfrac{(\tau_k - \mu_k)[(\mu_k - 1)\widetilde{x}_k + 2\nu_k \overline{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\widetilde{x}_k > (\mu_k - 1 + 2\nu_k)\overline{\Gamma}_{k-1}, \end{cases}$$

$$\overline{\Gamma}_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\lambda_0 \nu_0}{2a_0}\right).$$

Scheme Illustration of the Second Revised Policy



(a) $\mu_k > 0$ (b) $\mu_k < 0$

Figure: The scheme of the second revised policy

Performance

- Denote $\Delta\widetilde{x}_k = \widetilde{x}_k - \check{x}_k$.
- The second type of revised policies achieves the same total mean as the pre-committed optimal mean-variance policy of the T-period problem (MV) does, while having smaller total variance than the pre-committed optimal policy does, i.e.,

$$E(\widetilde{x}_T + \sum_{j=0}^{T-1} \Delta\widetilde{x}_j | x_0)|_{\check{\mathbf{u}}^*} = E(x_T|x_0)|_{\mathbf{u}^*},$$

$$Var(\widetilde{x}_T + \sum_{j=0}^{T-1} \Delta\widetilde{x}_j | x_0)|_{\check{\mathbf{u}}^*} < Var(x_T|x_0)|_{\mathbf{u}^*}.$$

Outline

Conclusion

- The dynamic mean-variance portfolio selection in markets with all risky assets is not time consistent in efficiency, due to the inherent nonseparable nature of the involved variance term.
- By adding the initial investment level into the objective space, the concept of pseudo efficiency (type 1 or type 2) has been introduced.
- By relaxing the self-financing constraint, two revised policies have been proposed to tackle pseudo efficiency (type 1 or type 2), thus achieving better performance than the original dynamic mean-variance policy.

Thank you for your attention!