Time-Consistent Mean-Variance Portfolio Selection in Discrete and Continuous Time

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Mean-variance portfolio selection in one period

- Harry Markowitz (Portfolio selection, 1952):
  - maximise return and minimise risk
  - return=expectation
  - risk=variance
- Mean-variance portfolio selection with risk aversion $\gamma > 0$ in one period:

$$U(\vartheta) = E[x + \vartheta^\top \Delta S] - \frac{\gamma}{2}\mathrm{Var}[x + \vartheta^\top \Delta S] = \max_{\vartheta}!$$

- Solution is the so-called **mean-variance efficient strategy**, i.e.

$$\widetilde{\vartheta} := \frac{1}{\gamma}\mathrm{Cov}[\Delta S|\mathcal{F}_0]^{-1} E[\Delta S|\mathcal{F}_0] =: \widehat{\vartheta}.$$

- Question: How does this extend to multi-period or continuous time?

Basic problem

- **Markowitz problem:**

$$U(\vartheta) = E\left[x + \int_0^T \vartheta_u dS_u\right] - \frac{\gamma}{2}\,\mathrm{Var}\left[x + \int_0^T \vartheta_u dS_u\right] = \max_{(\vartheta_s)_{0\leq s\leq T}}!$$

- Static: criterion at time 0 determines optimal $\widetilde{\vartheta}$ via $\widetilde{g} = \int_0^T \widetilde{\vartheta} dS$.
- **Question:** more explicit dynamic description of $\widetilde{\vartheta}$ on $[0, T]$ from $\widetilde{g}$?

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- **Question:** more explicit dynamic description of $\widetilde{\vartheta}$ on $[0, T]$ from $\widetilde{g}$?
- Dynamic: Use $\widetilde{\vartheta}$ on $(0, t]$ and determine optimal strategy on $(t, T]$ via

$$U_t(\vartheta) = E\left[x + \int_0^T \vartheta_u dS_u \middle| \mathcal{F}_t\right] - \frac{\gamma}{2} \mathrm{Var}\left[x + \int_0^T \vartheta_u dS_u \middle| \mathcal{F}_t\right] = \max_{(\vartheta_s)_{t\le s\le T}} !$$

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- **Time-consistent mean-variance portfolio selection:**
Find a strategy $\widehat{\vartheta}$, which is "optimal" for $U_t(\vartheta)$ and time-consistent.

Previous literature

- Strotz (1956): "choose the best plan among those that [you] will actually follow." $\longrightarrow$ Recursive approach to time inconsistency for a different problem.

In Markovian models: Deterministic functions, HJB PDEs and verification thm.

- Ekeland et al. (2006): game theoretic formulation for different problems.
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2) Rigorous justification of the continuous-time formulation?

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Financial market:

- $\mathbb{R}^d$-valued semimartingale S wlog. $S = S_0 + M + A \in \mathcal{S}^2(P)$.
- $\Theta = \Theta_S := \{\vartheta \in L(S) \mid \int \vartheta dS \in \mathcal{S}^2(P)\} = L^2(M) \cap L^2(A)$.

Outline

1. Discrete time
2. Continuous time
3. Convergence of solutions

Local mean-variance efficiency in discrete time

- Use $x + \vartheta \cdot S := x + \int_0^T \vartheta_u dS_u = x + \sum_{i=1}^T \vartheta_i \Delta S_i$ and suppose $d = 1$.

Definition

A strategy $\widehat{\vartheta} \in \Theta$ is **locally mean-variance efficient (LMVE)** *if*

$$U_{k-1}(\widehat{\vartheta}) - U_{k-1}(\widehat{\vartheta} + \delta \mathbb{1}_{\{k\}}) \geq 0 \qquad P\text{-a.s.}$$

for all $k = 1, \dots, T$ and any $\delta = (\vartheta - \widehat{\vartheta}) \in \Theta$.

- Recursive optimisation (Källblad 2008): $\widehat{\vartheta} \in \Theta$ is LMVE if and only if

$$\widehat{\vartheta}_k = \frac{1}{\gamma} \frac{E[\Delta S_k | \mathcal{F}_{k-1}]}{\mathrm{Var}\left[\Delta S_k | \mathcal{F}_{k-1}\right]} - \frac{\mathrm{Cov}\left[\Delta S_k, \sum_{i=k+1}^T \widehat{\vartheta}_i \Delta S_i \middle| \mathcal{F}_{k-1}\right]}{\mathrm{Var}\left[\Delta S_k | \mathcal{F}_{k-1}\right]} = \frac{1}{\gamma}\lambda_k - \xi_k(\widehat{\vartheta})$$

for $k = 1, \dots, T$.

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$$= \frac{1}{\gamma} \frac{\Delta A_k}{E[(\Delta M_k)^2 | \mathcal{F}_{k-1}]} - \frac{E\left[\Delta M_k E\left[\sum_{i=k+1}^T \widehat{\vartheta}_i \Delta S_i \middle| \mathcal{F}_k\right] \middle| \mathcal{F}_{k-1}\right]}{E[(\Delta M_k)^2 | \mathcal{F}_{k-1}]}$$

for $k = 1, \dots, T$.

Structure condition and mean-variance tradeoff process

- S satisfies the **structure condition (SC)**, i.e. there exists a predictable process λ such that

$$A_k = \sum_{i=1}^{k} \lambda_i E\left[(\Delta M_i)^2 | \mathcal{F}_{i-1}\right] = \sum_{i=1}^{k} \lambda_i \Delta\langle M\rangle_i$$

  for $k = 0, \dots, T$ and the **mean-variance tradeoff process (MVT)**

$$K_k := \sum_{i=1}^{k} \frac{\left(E[\Delta S_i | \mathcal{F}_{i-1}]\right)^2}{\mathrm{Var}\left[\Delta S_i | \mathcal{F}_{i-1}\right]} = \sum_{i=1}^{k} \lambda_i^2 \Delta\langle M\rangle_i = \sum_{i=1}^{k} \lambda_i \Delta A_i$$

  for $k = 0, \dots, T$ is finite-valued, i.e. $\lambda \in L^2_{loc}(M)$.

- If the LMVE strategy $\widehat{\vartheta}$ exists, then $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.
- **Comments:** 1) SC and MVT also appear naturally in other quadratic optimisation problems in mathematical finance; see Schweizer (2001). 2) No arbitrage condition: $A \ll \langle M\rangle$.

Expected future gains

- For each $\vartheta \in \Theta$, define the **expected future gains** $Z(\vartheta)$ and the square integrable martingale $Y(\vartheta)$ by

$$
\begin{aligned}
Z_k(\vartheta) :&= E\left[\sum_{i=k+1}^{T} \vartheta_i \Delta S_i \middle| \mathcal{F}_k\right] = E\left[\sum_{i=1}^{T} \vartheta_i \Delta A_i \middle| \mathcal{F}_k\right] - \sum_{i=1}^{k} \vartheta_i \Delta A_i \\
&=: Y_k(\vartheta) - \sum_{i=1}^{k} \vartheta_i \Delta A_i \\
&= Y_0(\vartheta) + \sum_{i=1}^{k} \xi_i(\vartheta) \Delta M_i + L_k(\vartheta) - \sum_{i=1}^{k} \vartheta_i \Delta A_i
\end{aligned}
$$

for $k = 0, 1, \dots, T$ inserting the **GKW decomposition** of $Y(\vartheta)$.

Lemma

The LMVE strategy $\widehat{\vartheta}$ exists if and only if
1) S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$, and 2) $\widehat{\vartheta} = \frac{1}{\gamma}\lambda - \xi(\widehat{\vartheta})$.

Global description of $\xi(\widehat{\vartheta})$ via FS decomposition

- Combining both representations we obtain

$$\sum_{i=1}^{T} \widehat{\vartheta}_i \Delta A_i = \sum_{i=1}^{T} \left(\frac{1}{\gamma} \lambda_i - \xi_i(\widehat{\vartheta}) \right) \Delta A_i$$

$$= Y_0(\widehat{\vartheta}) + \sum_{i=1}^{T} \xi_i(\widehat{\vartheta}) \Delta M_i + L_T(\widehat{\vartheta})$$

$$\frac{1}{\gamma} K_T = \frac{1}{\gamma} \sum_{i=1}^{T} \lambda_i \Delta A_i = Y_0(\widehat{\vartheta}) + \sum_{i=1}^{T} \xi_i(\widehat{\vartheta}) \Delta S_i + L_T(\widehat{\vartheta}) \qquad (1)$$

- (1) is almost the **Föllmer–Schweizer (FS) decomposition** of $\frac{1}{\gamma} K_T$.
- The integrand $\xi(\widehat{\vartheta}) =: \frac{1}{\gamma} \widehat{\xi}$ in the FS decomposition yields the **locally risk-minimising strategy** for the contingent claim $\frac{1}{\gamma} K_T$.
- **Global description:** $\widehat{\vartheta} \in \Theta$ exists iff (1) and $\widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$.

Continuous time setting

- Increasing, integrable, predictable process B called "operational time" such that: $A = a \cdot B$, $\langle M, M \rangle = \widetilde{c}^M \cdot B$ and $a = \widetilde{c}^M \lambda + \eta$ with $\eta \in \mathrm{Ker}(\widetilde{c}^M)$.
- S satisfies the **structure condition (SC)**, if $\eta = 0$, i.e.

$$A = \int d\langle M \rangle \lambda,$$

and the **mean-variance tradeoff process (MVT)**

$$K_t := \int_0^t \lambda_u^\top d\langle M \rangle_u \lambda_u = \int_0^t \lambda_u dA_u < +\infty.$$

- **Expected future gains** $Z(\vartheta)$ and **GKW decomposition** of $Y(\vartheta)$

$$\begin{aligned} Z_t(\vartheta) :&= E\left[\int_t^T \vartheta_u dS_u \middle| \mathcal{F}_t \right] = E\left[\int_0^T \vartheta_u dA_u \middle| \mathcal{F}_t \right] - \int_0^t \vartheta_u dA_u \\ &=: Y_t(\vartheta) - \int_0^t \vartheta_u dA_u \\ &= Y_0(\vartheta) + \int_0^t \xi_u(\vartheta) dM_u + L_t(\vartheta) - \int_0^t \vartheta_u dA_u \end{aligned}$$

Local mean-variance efficiency in continuous time

- Idea: Combine **recursive optimisation** with a **limiting argument**.

Definition

A strategy $\widehat{\vartheta} \in \Theta$ is **locally mean-variance efficient** *(in continuous time) if*

$$\liminf_{n\to\infty} u^{\Pi_n}[\widehat{\vartheta}, \delta] := \liminf_{n\to\infty} \sum_{t_i, t_{i+1} \in \Pi_n} \frac{U_{t_i}(\widehat{\vartheta}) - U_{t_i}(\widehat{\vartheta} + \delta \mathbb{1}_{(t_i, t_{i+1}]})}{E[B_{t_{i+1}} - B_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]} \geq 0 \quad P \otimes B\text{-a.e.}$$

for any increasing sequence (Π_n) of partitions such that $|\Pi_n| \to 0$ and any $\delta \in \Theta$.

- Inspired by the concept of local risk-minimisation (LRM); Schweizer (88, 08).

$$\lim_{n\to\infty} u^{\Pi_n}[\widehat{\vartheta}, \delta] = \left(\gamma\left(\xi(\widehat{\vartheta}) + \widehat{\vartheta}\right) - \lambda + \frac{\gamma}{2}\delta\right)^{\top} c^M \delta - \delta^{\top} \eta \qquad P \otimes B\text{-a.e.}$$

- **Remarks:** 1) Convergence without any additional assumptions, i.e. boundedness assumptions on δ and continuity of A.
  2) Generalises also results for LRM.

The LMVE strategy $\widehat{\vartheta}$ in continuous time

Theorem

1) The LMVE strategy $\widehat{\vartheta} \in \Theta$ exists if and only if

i) *S satisfies (SC) with $\lambda \in L^2(M)$, i.e. $K_T \in L^1(P)$.*

ii) *$\widehat{\vartheta} = \frac{1}{\gamma}\lambda - \xi(\widehat{\vartheta})$, i.e. $\widehat{J}(\widehat{\vartheta}) = \widehat{\vartheta}$, where $\widehat{J}(\psi) := \frac{1}{\gamma}\lambda - \xi(\psi)$ for $\psi \in \Theta$ and $\xi(\psi)$ is the integrand in the GKW decomposition of $\int_0^T \psi_u dA_u$.*

2) If K is bounded and continuous, $\widehat{J}(\cdot)$ is a contraction on $(\Theta, \|.\|_{\beta,\infty})$ where

$$\|\vartheta\|_{\beta,\infty} := \left\| \left(\int_0^T \frac{1}{\mathcal{E}(-\beta K)_u} \vartheta_u^\top d\langle M\rangle_u \vartheta_u \right)^{\frac{1}{2}} \right\|_{L^2(P)} \sim \|\vartheta\|_{L^2(M)} + \|\vartheta\|_{L^2(A)}.$$

In particular, the LMVE strategy $\widehat{\vartheta}$ is given as the limit $\widehat{\vartheta} = \lim_{n\to\infty} \vartheta^n$ in $(\Theta, \|.\|_{\beta,\infty})$, where $\vartheta^{n+1} = \widehat{J}(\vartheta^n)$ for $n \geq 1$, for any $\vartheta^0 = \vartheta \in \Theta$.

Global description of $\xi(\widehat{\vartheta})$ via FS decomposition

Theorem

The LMVE strategy $\widehat{\vartheta} \in \Theta$ exists if and only if S satisfies (SC) and the MVT process $K_T \in L^1(P)$ and can be written as

$$K_T = \widehat{K}_0 + \int_0^T \widehat{\xi} dS + \widehat{L}_T \tag{2}$$

with $\widehat{K}_0 \in L^2(\mathcal{F}_0)$, $\widehat{\xi} \in L^2(M)$ such that $\widehat{\xi} - \lambda \in L^2(A)$ and $\widehat{L} \in \mathcal{M}_0^2(P)$ strongly orthogonal to M. In that case, $\widehat{\vartheta} = \frac{1}{\gamma}(\lambda - \widehat{\xi})$, $\xi(\widehat{\vartheta}) = \frac{1}{\gamma}\widehat{\xi}$ and $U(\widehat{\vartheta}) = \dots$ (2).

- If the **minimal martingale measure** exists, i.e. $\frac{d\widehat{P}}{dP} := \mathcal{E}(-\lambda \cdot M)_T \in L^2(P)$ and strictly positive, and $K_T \in L^2(P)$, then

$$Z_t(\widehat{\vartheta}) = \frac{1}{\gamma}\left(\widehat{K}_0 + \int_0^t \widehat{\xi} dS + \widehat{L}_t - K_t\right) = \frac{1}{\gamma}\widehat{E}[K_T - K_t | \mathcal{F}_t],$$

  and $\widehat{\xi}$ is related to the GKW of K_T under $\widehat{P}$; see Choulli et al. (2010).

- Application in concrete models: 1) λ, 2) K, 3) $\mathcal{E}(-\lambda \cdot M)$ and 4) $\widehat{\xi}$ ...

Discretisation of the financial market

- Let $(\Pi_n)_{n\in\mathbb{N}}$ increasing such that $|\Pi_n| \to 0$ and $S = S_0 + M + A$.

Discretisation of processes

- $S^n_t := S_{t_i}$, $M^n_t := M_{t_i}$ and $A^n_t := A_{t_i}$ for $t \in [t_i, t_{i+1})$ and all $t_i \in \Pi_n$.

Discretisation of filtration

- $\mathcal{F}^n_{t_i} := \mathcal{F}_{t_i}$ for $t \in [t_i, t_{i+1})$ and all $t_i \in \Pi_n$ and $\mathbb{F}^n := (\mathcal{F}^n_t)_{0\leq t\leq T}$.

Canonical decomposition of $S^n = S_0 + \bar{M}^n + \bar{A}^n \in \mathcal{S}^2(P, \mathbb{F}^n)$

- $\bar{A}^n_t := \sum_{k=1}^{i} E[\Delta A^n_{t_k} | \mathcal{F}_{t_{k-1}}] = A^n_t - \mathbf{M^{A,n}_t}$
- $\bar{M}^n_t := M^n_t + \mathbf{M^{A,n}_t}$ for $t \in [t_i, t_{i+1})$

where the **"discretisation error"** is given by the $\mathbb{F}^n$-martingale

$$\mathbf{M^{A,n}_t} := \sum_{k=1}^{i} (\Delta A^n_{t_k} - E[\Delta A^n_{t_k} | \mathcal{F}_{t_{k-1}}]) \quad \text{for } t \in [t_i, t_{i+1}).$$

Convergence of solutions $\widehat{\vartheta}^n$

- Due to time inconsistency usual abstract arguments don't work.
- Work with **global description** directly to show

$$\widehat{\vartheta}^n = \frac{1}{\gamma}\big(\lambda^n - \widehat{\xi}^n\big) \overset{L^2(M)}{\longrightarrow} \widehat{\vartheta} = \frac{1}{\gamma}\big(\lambda - \widehat{\xi}\big), \quad \text{as } |\Pi^n| \to 0.$$

- **Discrete-** and **continuous-time FS decomposition**

$$K_T^n = \widehat{K}_0^n + \sum_{t_i \in \Pi_n} \widehat{\xi}_{t_i}^n \Delta S_{t_i}^n + \widehat{L}_T^n \quad \text{and} \quad K_T = \widehat{K}_0 + \int_0^T \widehat{\xi}_u dS_u + \widehat{L}_T.$$

- For this we establish
  1) $\lambda^n = \sum_{t_i, t_{i+1} \in \Pi_n} \frac{\Delta \bar{A}^n_{t_{i+1}}}{E[(\Delta \bar{M}^n_{t_{i+1}})^2 | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i, t_{i+1}]} \overset{L^2(M)}{\longrightarrow} \lambda$
  2) $K_T^n = \sum_{t_i, t_{i+1} \in \Pi_n} \lambda^n_{t_{i+1}} \Delta \bar{A}^n_{t_{i+1}} \overset{L^2(P)}{\longrightarrow} K_T = \int_0^T \lambda_u dA_u$
  3) 2), $|\Pi_n| \to 0$ implies $\widehat{\xi}^n \overset{L^2(M)}{\longrightarrow} \widehat{\xi}$.
- **Problem** to control the **"discretisation error" $\mathbf{M^{A,n}}$**.
- **Simple sufficient condition:** $K = \int \frac{dK}{dt} dt$ and $\frac{dK}{dt}$ uniformly bounded.

Some references

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Thank you for your attention!

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