# Optimal investment under relative performance concerns

## GE.Espinosa Joint work with N.Touzi

June 26th, 2010

GE.Espinosa Joint work with N.Touzi [Optimal investment under relative performance concerns](#page-48-0)

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- $\triangleright$  Classical portfolio optimization: maximization of one's utility with respect to one's personal wealth or consumption
- $\triangleright$  Economical literature: relative wealth concerns

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- $\triangleright$  Classical portfolio optimization: maximization of one's utility with respect to one's personal wealth or consumption
- $\blacktriangleright$  Economical literature: relative wealth concerns

Aim: Try to derive a portfolio optimization theory with such relative wealth concerns.

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The market:

- a non-risky asset with 0 interest rate
- a d-dimensional risky asset S
- N agents

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The market:

- a non-risky asset with 0 interest rate
- a d-dimensional risky asset S
- N agents

The dynamics of  $S$  is given by:

$$
dS_t = \text{diag}(S_t)\sigma_t(\theta_t dt + dW_t)
$$

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 $\sigma$  is assumed to be symmetric definite.

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 $A \oplus A$  and  $A \oplus A$ 

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We will first assume that all agents are similar.

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We write  $X^i$  the wealth process of agent  $i$  and  $\pi^i$  the portfolio of agent i. Investment horizon T.

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We write  $X^i$  the wealth process of agent  $i$  and  $\pi^i$  the portfolio of agent i. Investment horizon T.

Characteristics of agent i:

- exponential utility function with risk preference  $\eta > 0$
- relative performance preference  $\lambda \in [0,1]$
- average wealth of other agents  $\bar{X}^i = \frac{1}{N}$  $N-1$  $\sum$ j≠i X j

 $A \cap B$   $A \cap A \subseteq B$   $A \subseteq B$ 

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Thus agent  $i$  wants to maximize upon admissible  $\pi^i$ :

$$
-\mathbb{E}e^{-\frac{1}{\eta}[(1-\lambda)X^i_T+\lambda(X^i_T-\bar{X}^i_T)]}
$$

given other  $\pi^j$   $(j \neq i)$ 

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By symmetry, at the equilibrium, it is the same as:

$$
\sup_{\pi^i}-\mathbb{E}e^{-\frac{1-\lambda}{\eta}X^i_T}
$$

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Same as in the classical case but  $\eta \to \frac{\eta}{1-\lambda}$ 

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\sup_{\pi^i}-\mathbb{E}e^{-\frac{1-\lambda}{\eta}X^i_T}
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Same as in the classical case but  $\eta \to \frac{\eta}{1-\lambda}$ 

So the optimal portfolio is (for deterministic  $\theta$ ,  $\lambda$  < 1):

$$
\hat{\pi}_t^i = \frac{\eta}{(1-\lambda)} \sigma_t^{-1} \theta_t
$$

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### Influence of  $\lambda$ :

- $|\hat{\pi}^i|$  is increasing w.r.t  $\lambda$
- if  $\lambda \rightarrow 1$ ,  $|\hat{\pi}^i| \rightarrow \infty$  a.s.

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Specific parameters:

- risk preference  $\eta_i > 0$
- relative performance preference  $\lambda_i \in [0,1]$

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Specific parameters:

- risk preference  $\eta_i > 0$
- relative performance preference  $\lambda_i \in [0,1]$

Portfolio constraints:

Each agent has an area of investment.  $\pi^i$  must stay in a certain  $A_i$ that will be assumed to be a vector sub-space of  $\mathbb{R}^d$ .

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So finally agent *i's* criterion:

$$
\sup_{\pi^i \in \mathcal{A}_i} -\mathbb{E}e^{-\frac{1}{\eta_i}[X^{i,\pi^i}_T - \lambda_i\bar{X}^i_T]}
$$

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$$

And we look for Nash equilibria between the N agents.

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目



Using the ideas of El Karoui-Rouge or Hu-Imkeller-Muller for optimal investment in incomplete markets, we derive a (quadratic) BSDE:

$$
dY_t^i = \left(\frac{\eta|\theta_t|^2}{2} - \frac{1}{2\eta} |Z_t^i + \eta \theta_t - P_{\sigma_t A_i} (Z_t^i + \eta \theta_t)|^2\right) dt + Z_t^i . dB_t
$$
  

$$
Y_T^i = \lambda(\bar{X}_T^i - \bar{x}_i) = \frac{\lambda}{N-1} \sum_{j \neq i} \int_0^T \pi_u^j . \sigma_u dB_u
$$

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$$

And the optimal portfolio is given by:

$$
\sigma_t \hat{\pi}_t^i = P_{\sigma_t A_i} (Z_t^i + \eta \theta_t)
$$

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Putting them together it brings:

$$
Y_0^i = -\eta \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta} \int_0^T |\mathcal{Q}_t^i(Z_t^i)|^2 dt - \int_0^T (Z_t^i - \frac{\lambda}{N-1} \sum_{j \neq i} P_t^j(Z_t^j)). dB_t
$$

where  $P_i =$  orthogonal projection on  $\sigma A_i$ ,  $Q_i = I - P_i$ ,  $\mathbb{Q} =$  the martingale probability and  $B$  a Brownian motion under  $\mathbb{Q}$ .

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It can be rewritten as:

$$
Y_0^i = -\eta \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta} \int_0^T |\mathsf{Q}_t^i([\psi_t(\zeta_t)]^i)|^2 dt - \int_0^T \zeta_t^i dB_t
$$
  
where  $Y \in \mathbb{R}^N$ ,  $\zeta \in M_{N,d}(\mathbb{R})$  and  $\psi \in GL(M_{N,d}(\mathbb{R}))$ .

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 $\rightarrow$  N-dimensional system of coupled quadratic BSDEs.

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Assume  $\sigma$  and  $\theta$  are deterministic:

**Theorem:** There exists a Nash equilibrium and the equilibrium portfolio for agent  $i$  is given by:

$$
\pi^{i} = \eta \sigma^{-1} P^{i} \left[ I - \frac{\frac{\lambda}{N-1}}{1 + \frac{\lambda}{N-1}} \sum_{j \neq i} P^{j} \left( I + \frac{\lambda}{N-1} P^{i} \right) \right]^{-1} \theta
$$

 $(P^i$  is the orthogonal projection on  $\sigma {\cal A}_i)$ 

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$$

 $(P^i$  is the orthogonal projection on  $\sigma {\cal A}_i)$ 

Under the assumption:

$$
\lambda < 1 \text{ or } \bigcap_{i=1}^N A_i = \{0\}
$$

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If d is fixed:

#### **Theorem:** Let d be fixed, and assume moreover that 1 N  $\sum$ N  $i=1$  $P^i \to U$  in  $\mathcal{L}(\mathbb{R}^d)$  with  $||\lambda U|| < 1$ . Then  $\pi^i_\mathcal{N} \to \pi^i_\infty$  where:

$$
\pi^i_\infty = \eta \sigma^{-1} P^i [(I - \lambda U)^{-1} \theta]
$$

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Market index: 
$$
\bar{X}_t^N = \frac{1}{N} \sum_{i=1}^N X_t^i
$$
  
We find:

$$
d\bar{X}_t^{\infty} = \eta U(I - \lambda U)^{-1} \theta_t . [\theta_t dt + dW_t]
$$

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Market index: 
$$
\bar{X}_t^N = \frac{1}{N} \sum_{i=1}^N X_t^i
$$
  
We find:

$$
d\bar{X}^{\infty}_t = \eta U(I - \lambda U)^{-1} \theta_t \cdot [\theta_t dt + dW_t]
$$

Moreover,  $U(I - \lambda U)^{-1}$  is diagonalizable with eigenvalues

$$
0<\frac{\mu_1}{1-\lambda\mu_1}<...<\frac{\mu_d}{1-\lambda\mu_d}<1
$$

and with the same orthonormal eigenvectors as  $U$  (independent of  $\lambda$ ).

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 $\rightarrow$  Both the drift and the risk (volatility) of the market increase with  $\lambda$ 

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- $\rightarrow$  Both the drift and the risk (volatility) of the market increase with  $\lambda$
- $\rightarrow$  encourages financial bubbles

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Each agent can invest in the whole market:

$$
\forall i, \ A_i = \mathbb{R}^d
$$

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Each agent can invest in the whole market:

$$
\forall i, \ A_i = \mathbb{R}^d
$$

Under the assumption 
$$
\prod_{j=1}^{N} \lambda_j < 1
$$
, there is an equilibrium.

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First case:  $\forall i, \lambda_i = \lambda$ , then:

$$
\hat{\pi}_t^i = \left[ \frac{N-1}{N+\lambda-1} + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)} \frac{\eta^N}{\eta_i} \right] \pi_t^{0,i}
$$

 $\eta^\mathcal{N}$  is the average of the  $\eta_j$ 's.

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As  $N\to\infty$ , if  $\eta^N\to\eta>0$  then the equilibrium portfolio of agent i converges to:

$$
\hat{\pi}^{\infty,i}_t=\left(1+\frac{\lambda}{1-\lambda}\frac{\eta}{\eta_i}\right)\pi^{0,i}_t
$$

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As  $N\to\infty$ , if  $\eta^N\to\eta>0$  then the equilibrium portfolio of agent i converges to:

$$
\hat{\pi}^{\infty,i}_t=\left(1+\frac{\lambda}{1-\lambda}\frac{\eta}{\eta_i}\right)\pi^{0,i}_t
$$

Same conclusions as in the beginning.

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Second case: 
$$
\forall j \neq i_0, \lambda_j = 1, \lambda_{i_0} < 1 \ (\forall i, \eta_i = \eta)
$$
, then:

$$
\hat{\pi}^{i_0}_t = \left[\frac{1}{1-\lambda_{i_0}} + \frac{\lambda_{i_0}(N-1)}{1-\lambda_{i_0}}\right] \pi^0_t
$$

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Second case: 
$$
\forall j \neq i_0, \lambda_j = 1, \lambda_{i_0} < 1 \ (\forall i, \eta_i = \eta)
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\hat{\pi}^{i_0}_t = \left[\frac{1}{1-\lambda_{i_0}} + \frac{\lambda_{i_0}(N-1)}{1-\lambda_{i_0}}\right] \pi^0_t
$$

As  $N\to\infty$ , even if  $\lambda_{j_0} < 1$ ,  $|\pi_t^{j_0}| \to \infty$  a.s (except for  $\lambda_{j_0} = 0).$ 

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Second case: 
$$
\forall j \neq i_0, \lambda_j = 1, \lambda_{i_0} < 1 \ (\forall i, \eta_i = \eta)
$$
, then:

$$
\hat{\pi}_t^{i_0} = \left[\frac{1}{1-\lambda_{i_0}} + \frac{\lambda_{i_0}(N-1)}{1-\lambda_{i_0}}\right] \pi_t^0
$$

As  $N\to\infty$ , even if  $\lambda_{j_0} < 1$ ,  $|\pi_t^{j_0}| \to \infty$  a.s (except for  $\lambda_{j_0} = 0).$ 

 $\rightarrow$  Impact of surrounding "stupidity".

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- 
$$
d = N
$$
,  $A_i = \mathbb{R}e_i$ 

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$$
d = N, A_i = \mathbb{R}e_i
$$
  
-  $\sigma^2 = \sigma^2 \begin{pmatrix} 1 & \rho^2 \\ \cdot & \cdot \\ \rho^2 & 1 \end{pmatrix}$  with  $\rho \in (-1, 1)$  and  $\sigma > 0$ 

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$$
d = N, A_i = \mathbb{R}e_i
$$
  
-  $\sigma^2 = \sigma^2 \begin{pmatrix} 1 & \rho^2 \\ \cdot & \cdot \\ \rho^2 & 1 \end{pmatrix}$  with  $\rho \in (-1, 1)$  and  $\sigma > 0$ 

- we also assume  $\forall i, \theta_i = \theta$ .

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As  $N \rightarrow \infty$  we find:

$$
\hat{\pi}^i = \frac{\eta \theta}{\sigma} \frac{1}{1 - \lambda \rho^2} e_i
$$

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As  $N \rightarrow \infty$  we find:

$$
\hat{\pi}^i = \frac{\eta \theta}{\sigma} \frac{1}{1-\lambda \rho^2} e_i
$$

So:

- the more you look at other agents ( $\lambda$  close to 1)
- the more correlated the assets are  $(\rho^2$  close to  $1)$ the more risk you take.

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$$

So:

- the more you look at other agents ( $\lambda$  close to 1)
- the more correlated the assets are  $(\rho^2$  close to  $1)$ the more risk you take.

For independent investments ( $\rho = 0$ ), we find the classical optimal portfolio: no impact of  $\lambda$ .

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- Here again  $d=N$ . But  $A_i=(\mathbb{R}e_i)^{\perp}$ 

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- Here again  $d=N$ . But  $A_i=(\mathbb{R}e_i)^{\perp}$
- $-\sigma = \sigma I$  and  $\forall i, \theta_i = \theta$ .

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We find:

$$
\hat{\pi}^i_t = \frac{\eta \theta}{\sigma} \frac{1}{1 - \lambda + \frac{\lambda}{N-1}} \sum_{j \neq i} e_j
$$

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We find:

$$
\hat{\pi}^i_t = \frac{\eta \theta}{\sigma} \frac{1}{1-\lambda+\frac{\lambda}{N-1}} \sum_{j \neq i} e_j
$$

Same kind of conclusions as for investment on the whole market, but smaller impact of  $\lambda$ , especially for small N.

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# Special thanks to J.Lebuchoux - Reech Aim

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