Optimal investment under relative performance concerns

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Joint work with N.Touzi

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- Classical portfolio optimization: maximization of one's utility with respect to one's personal wealth or consumption
- Economical literature: relative wealth concerns

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Aim: Try to derive a portfolio optimization theory with such relative wealth concerns.

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Characteristics of agent i:

- exponential utility function with risk preference $\eta > 0$
- relative performance preference $\lambda \in [0, 1]$
- average wealth of other agents $\bar{X}^i = \frac{1}{N-1} \sum_{j \neq i} X^j$

Thus agent i wants to maximize upon admissible π^i:

$$-\mathbb{E}e^{-\frac{1}{\eta}[(1-\lambda)X_T^i+\lambda(X_T^i-\bar{X}_T^i)]}$$

given other π^j $(j \neq i)$

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So the optimal portfolio is (for deterministic θ, $\lambda < 1$):

$$\hat{\pi}_t^i = \frac{\eta}{(1-\lambda)}\sigma_t^{-1}\theta_t$$

Influence of λ:

- $|\hat{\pi}^i|$ is increasing w.r.t λ
- if $\lambda \to 1$, $|\hat{\pi}^i| \to \infty$ a.s.

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Portfolio constraints:
Each agent has an area of investment. π^i must stay in a certain A_i that will be assumed to be a vector sub-space of $\mathbb{R}^d$.

So finally agent i's criterion:

$$\sup_{\pi^i \in \mathcal{A}_i} -\mathbb{E} e^{-\frac{1}{\eta_i}[X_T^{i,\pi^i} - \lambda_i \bar{X}_T^i]}$$

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And we look for Nash equilibria between the N agents.

Using the ideas of El Karoui-Rouge or Hu-Imkeller-Muller for optimal investment in incomplete markets, we derive a (quadratic) BSDE:

$$dY_t^i = \left(\frac{\eta|\theta_t|^2}{2} - \frac{1}{2\eta} \left| Z_t^i + \eta\theta_t - P_{\sigma_t A_i}(Z_t^i + \eta\theta_t) \right|^2 \right) dt + Z_t^i . dB_t$$

$$Y_T^i = \lambda(\bar{X}_T^i - \bar{x}_i) = \frac{\lambda}{N-1} \sum_{j \neq i} \int_0^T \pi_u^j . \sigma_u dB_u$$

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And the optimal portfolio is given by:

$$\sigma_t \hat{\pi}_t^i = P_{\sigma_t A_i}(Z_t^i + \eta\theta_t)$$

Putting them together it brings:

$$Y_0^i = -\eta \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta} \int_0^T |Q_t^i(Z_t^i)|^2 dt - \int_0^T (Z_t^i - \frac{\lambda}{N-1} \sum_{j \neq i} P_t^j(Z_t^j)).dB_t$$

where P_i = orthogonal projection on σA_i, $Q_i = I - P_i$, $\mathbb{Q}$ = the martingale probability and B a Brownian motion under $\mathbb{Q}$.

It can be rewritten as:

$$Y_0^i = -\eta \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{1}{2\eta} \int_0^T |Q_t^i([\psi_t(\zeta_t)]^i)|^2 dt - \int_0^T \zeta_t^i . dB_t$$

where $Y \in \mathbb{R}^N$, $\zeta \in M_{N,d}(\mathbb{R})$ and $\psi \in GL(M_{N,d}(\mathbb{R}))$.

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$\rightarrow$ N-dimensional system of coupled quadratic BSDEs.

Assume σ and θ are deterministic:

Theorem: There exists a Nash equilibrium and the equilibrium portfolio for agent i is given by:

$$\pi^i = \eta\sigma^{-1} P^i \left[I - \frac{\frac{\lambda}{N-1}}{1+\frac{\lambda}{N-1}} \sum_{j \neq i} P^j \left(I + \frac{\lambda}{N-1} P^i \right) \right]^{-1} \theta$$

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Under the assumption:

$$\lambda < 1 \text{ or } \bigcap_{i=1}^{N} A_i = \{0\}$$

If d is fixed:

Theorem: Let d be fixed, and assume moreover that $\frac{1}{N}\sum_{i=1}^{N} P^i \to U$ in $\mathcal{L}(\mathbb{R}^d)$ with $||\lambda U|| < 1$. Then $\pi^i_N \to \pi^i_\infty$ where:

$$\pi^i_\infty = \eta\sigma^{-1}P^i[(I-\lambda U)^{-1}\theta]$$

Market index: $\bar{X}_t^N = \frac{1}{N}\sum_{i=1}^{N} X_t^i$

We find:

$$d\bar{X}_t^\infty = \eta U(I - \lambda U)^{-1}\theta_t.[\theta_t dt + dW_t]$$

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Moreover, $U(I - \lambda U)^{-1}$ is diagonalizable with eigenvalues

$$0 < \frac{\mu_1}{1 - \lambda\mu_1} < ... < \frac{\mu_d}{1 - \lambda\mu_d} < 1$$

and with the same orthonormal eigenvectors as U (independent of λ).

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$\longrightarrow$ encourages financial bubbles

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Under the assumption $\prod_{j=1}^{N} \lambda_j < 1$, there is an equilibrium.

First case: $\forall i,\ \lambda_i = \lambda$, then:

$$\hat{\pi}_t^i = \left[\frac{N-1}{N+\lambda-1} + \frac{\lambda N}{(1-\lambda)(N+\lambda-1)} \frac{\eta^N}{\eta_i} \right] \pi_t^{0,i}$$

η^N is the average of the η_j's.

As $N \to \infty$, if $\eta^N \to \eta > 0$ then the equilibrium portfolio of agent i converges to:

$$\hat{\pi}_t^{\infty,i} = \left(1 + \frac{\lambda}{1-\lambda}\frac{\eta}{\eta_i}\right)\pi_t^{0,i}$$

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Same conclusions as in the beginning.

Second case: $\forall j \neq i_0,\ \lambda_j = 1,\ \lambda_{i_0} < 1\ (\forall i,\ \eta_i = \eta)$, then:

$$\hat{\pi}_t^{i_0} = \left[\frac{1}{1-\lambda_{i_0}} + \frac{\lambda_{i_0}(N-1)}{1-\lambda_{i_0}} \right] \pi_t^0$$

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$\to$ Impact of surrounding "stupidity".

- $d = N$, $A_i = \mathbb{R}e_i$

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- we also assume $\forall i,\ \theta_i = \theta$.

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So:

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- the more correlated the assets are (ρ^2 close to 1)

the more risk you take.

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For independent investments ($\rho = 0$), we find the classical optimal portfolio: no impact of λ.

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- $\sigma = \sigma I$ and $\forall i,\ \theta_i = \theta$.

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Same kind of conclusions as for investment on the whole market, but smaller impact of λ, especially for small N.

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