Dynamic Hedging of Conditional Value-at-Risk

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Alexander Melnikov melnikov@ualberta.ca

University of Alberta

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Introduction

In this talk, the problem of partial hedging is studied by constructing hedging strategies that minimize conditional value-at-risk (CVaR) of the portfolio. Two aspects of the problem are considered: minimization of CVaR with initial capital bounded from above, and minimization of hedging costs subject to a CVaR constraint. The Neyman-Pearson lemma is used to deduce semi-explicit solutions. The results are illustrated by constructing CVaR-efficient hedging strategies for a call option in the Black-Scholes model, call option in regime-switching telegraph market model and embedded call option for equity-linked life insurance contract.

Partial Hedging

- In a complete unconstrained financial market every contingent claim with discounted payoff H can be hedged perfectly.
- Perfect hedging requires initial capital in the amount of $H_0 = \mathbb{E}_{\mathbb{P}^*}[H]$.
- In a constrained market perfect hedging is not always possible.
- Example of a constraint: initial capital bounded by $\tilde{V}_0 < H_0$.
- The problem is to select the "best" partial hedging strategy.
- One of the approaches is to optimize a risk measure.

Selecting Target Risk Measure

- Properties of the optimal hedging strategy depend on the risk measure being optimized.
- Poor choice of the risk measure generally leads to poor results.
- Examples of risk measures:
  - Linear shortfall risk
  - Quadratic loss
  - Probability of successful hedging
  - Value-at-risk
  - Conditional value-at-risk
  - Lower/upper tail conditional expectation
  - Worst conditional expectation
  - Expected shortfall

Choosing a Risk Measure

Linear Shortfall Risk, Quadratic Loss, Probability of Successful Hedging

- Let random variable L represent loss (can be negative).
- **Linear shortfall risk**: $\mathbb{E}_{\mathbb{P}}[L^+]$,where $x^+ = \max(x, 0)$.
- **Quadratic loss**: $\mathbb{E}_{\mathbb{P}}[L^2]$.
- **Probability of successful hedging**: $\mathbb{P}(L \leq 0)$.

Choosing a Risk Measure

Value-at-Risk and Conditional Value-at-Risk

- VaR and CVaR are defined for a fixed level $\alpha \in (0,1)$.
- Let $L_{(\alpha)}$ and $L^{(\alpha)}$ be lower and upper α-quantiles of L:

$$
\begin{aligned}
L_{(\alpha)} &= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}[L \leq x] \geq \alpha\}, \\
L^{(\alpha)} &= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}[L \leq x] > \alpha\}
\end{aligned}
$$

- **Value-at-risk (VaR)** at level α:

$$
\mathrm{VaR}^{\alpha}(L) = L^{(1-\alpha)}.
$$

- **Conditional value-at-risk (CVaR)** at level α:

$$
\mathrm{CVaR}^{\alpha}(L) = \inf\left\{z + \tfrac{1}{\alpha} \cdot \mathbb{E}_{\mathbb{P}}\left[(L-z)^{+}\right] \ : \ z \in \mathbb{R}\right\}.
$$

- Note that the infimum in CVaR definition is always attained as minimum (see Rockafellar and Uryasev, 2000).

Choosing a Risk Measure

Tail Conditional Expectation, Worst Conditional Expectation and Expected Shortfall

- **Lower tail conditional expectation (lower TCE)** at level α:

$$\mathrm{TCE}_{\alpha}(L) = \mathbb{E}[L \mid L \geq L_{(1-\alpha)}].$$

- **Upper tail conditional expectation (upper TCE)** at level α:

$$\mathrm{TCE}^{\alpha}(L) = \mathbb{E}[L \mid L \geq L^{(1-\alpha)}].$$

- **Worst conditional expectation (WCE)** at level α:

$$\mathrm{WCE}_{\alpha}(L) = \sup \left\{ \mathbb{E}[L \mid A] \ : \ A \in \mathcal{F}, \mathbb{P}[A] > \alpha \right\}.$$

- **Expected shortfall (ES)** at level α:

$$\mathrm{ES}_{\alpha}(L) = \frac{1}{\alpha} \cdot \Big(\mathbb{E}[L \cdot \mathbf{1}_{\{L \geq L_{(1-\alpha)}\}}] + L_{(1-\alpha)} \cdot \Big(\mathbb{P}[L \geq L_{(1-\alpha)}] - \alpha \Big) \Big).$$

- The following relationships are true for any loss function:

$$\begin{aligned} \mathrm{ES}_\alpha &= \mathrm{CVaR}^\alpha, \\ \mathrm{TCE}^\alpha &\leq \mathrm{TCE}_\alpha \leq \mathrm{CVaR}^\alpha, \\ \mathrm{TCE}^\alpha &\leq \mathrm{WCE}_\alpha \leq \mathrm{CVaR}^\alpha. \end{aligned}$$

- $\mathrm{TCE}^\alpha(L) = \mathrm{TCE}_\alpha(L) = \mathrm{WCE}_\alpha(L) = \mathrm{CVaR}^\alpha(L)$ if and only if

$$\mathbb{P}(L \geq L^{(1-\alpha)}) = \alpha, \ \mathbb{P}(L > L_{(1-\alpha)}) > 0$$

or

$$\mathbb{P}(L \geq L^{(1-\alpha)}, \ L \neq L_{(1-\alpha)}) = 0.$$

- Consider a world with three states: $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 0.48$, $\mathbb{P}(\omega_3) = 0.04$ and three different loss functions: L_1, L_2 and L_3.

	ω_1	ω_2	ω_3	$\mathbb{P}[L \leq 0]$	$\mathrm{VaR}_{0.05}$	$\mathbb{E}[L^2]$	$\mathrm{CVaR}_{0.05}$
L_1	-1	1	10	0.48	1.00	4.96	8.20
L_2	-1	1	100	0.48	1.00	400.96	80.20
L_3	-2	1	10	0.48	1.00	6.40	8.20

- In the example above:
    - $\mathbb{P}[L_1 \leq 0] = \mathbb{P}[L_2 \leq 0] = \mathbb{P}[L_3 \leq 0]$,
    - $\mathrm{VaR}_{0.05}(L_1) = \mathrm{VaR}_{0.05}(L_2) = \mathrm{VaR}_{0.05}(L_3)$,
    - $\mathbb{E}[(L_1)^2] \leq \mathbb{E}[(L_3)^2] \leq \mathbb{E}[(L_2)^2]$,
    - $\mathrm{CVaR}_{0.05}(L_1) = \mathrm{CVaR}_{0.05}(L_3) \leq \mathrm{CVaR}_{0.05}(L_2)$.

- Let the discounted price process X_t be a semimartingale on a standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- A *self-financing strategy*: initial capital $V_0 > 0$ and a predictable process ξ_t. For each strategy (V_0, ξ) the value process V_t is

$$V_t = V_0 + \int_0^t \xi_s dX_s, \quad \forall t \in [0, T].$$

A strategy (V_0, ξ) is *admissible* if

$$V_t \geq 0, \quad \forall t \in [0, T], \quad \mathbb{P}-a.s.$$

Denote the set of all admissible self-financing strategies by $\mathcal{A}$.

- Consider a short position in a contingent claim whose discounted payoff is an $\mathcal{F}_T$–measurable random variable $H \in L^1(\mathbb{P})$, $H \geq 0$.
- In a complete market there exists a unique martingale measure $\mathbb{P}^* \approx \mathbb{P}$, and the perfect hedging strategy requires allocating initial capital $H_0 = \mathbb{E}_{\mathbb{P}^*}[H]$ (risk-neutral price).
- For each strategy (V_0, ξ) define loss function:

$$L = L(V_0, \xi) = H - V_T.$$

- Capital constraint: $V_0 \leq \tilde{V}_0 < H_0$.
- The problem is to minimize CVaR over the set of admissible self-financing strategies:

$$\begin{cases} \mathrm{CVaR}_\alpha(V_0, \xi) \longrightarrow \min\limits_{(V_0,\xi)\in\mathcal{A}}, \\ V_0 \leq \tilde{V}_0. \end{cases}$$

- Recall that

$$\mathrm{CVaR}^{\alpha}(V_0,\xi) = \inf\left\{ z + \frac{1}{\alpha}\cdot \mathbb{E}_{\mathbb{P}}\left[(H - V_T - z)^+\right] \ : \ z\in\mathbb{R}\right\},$$

and define

$$\begin{aligned} \mathcal{A}_{\tilde{V}_0} &= \{(V_0,\xi) \mid (V_0,\xi)\in\mathcal{A}, \quad V_0 \le \tilde{V}_0\}, \\ c(z) &= z + \frac{1}{\alpha}\cdot \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}_{\mathbb{P}}\left[(H - V_T - z)^+\right]. \end{aligned}$$

- Then

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_{\alpha}(V_0,\xi) = \min_{z\in\mathbb{R}} c(z).$$

- If we manage to derive an explicit form for $c(z)$, the initial problem is reduced to a problem of one-dimensional minimization.

- The problem is to find an explicit expression for the function

$$c(z) = z + \frac{1}{\alpha} \cdot \min_{(V_0,\xi) \in \mathcal{A}_{\tilde{V}_0}} \mathbb{E}_{\mathbb{P}} \left[(H - V_T - z)^+ \right].$$

- Note that $(H - V_T - z)^+ \equiv ((H - z)^+ - V_T)^+$and consider the problem

$$\mathbb{E}_{\mathbb{P}} \left[(H - z)^+ - V_T)^+ \right] \longrightarrow \min_{(V_0,\xi) \in \mathcal{A}_{\tilde{V}_0}}.$$

- The latter is a problem of linear shortfall risk minimization with respect to a contingent claim whose payoff $(H - z)^+$ depends on parameter z. The solution $(\hat{V}_0(z), \hat{\xi}(z))$ may be derived with the help of Neyman-Pearson lemma (Föllmer and Leukert, 2000).

The optimal strategy $(\hat{V}_0(z), \hat{\xi}(z))$ for the problem

$$\mathbb{E}_{\mathbb{P}}\left[(H-z)^+ - V_T)^+\right] \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}}$$

is a perfect hedge for $\tilde{H}(z) = (H-z)^+\tilde{\varphi}(z)$, where

$$\begin{aligned}
\tilde{\varphi}(z) &= \mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z)\right\}} + \gamma(z)\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z)\right\}}, \\
\tilde{a}(z) &= \inf\left\{a \geq 0 : \mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > a\right\}}\right] \leq \tilde{V}_0\right\}, \\
\gamma(z) &= \frac{\tilde{V}_0 - \mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z)\right\}}\right]}{\mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z)\right\}}\right]}.
\end{aligned}$$

The optimal strategy $(\hat{V}_0, \hat{\xi})$ for the problem

$$\mathrm{CVaR}_\alpha(V_0, \xi) \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}}$$

is a perfect hedge for $\tilde{H}(\hat{z}) = (H - \hat{z})^+ \tilde{\varphi}(\hat{z})$, where $\tilde{\varphi}(z)$ is the randomized test from linear shortfall risk subproblem, $\hat{z}$ is the point of global minimum of

$$c(z) = \begin{cases} z + \frac{1}{\alpha} \cdot \mathbb{E}_{\mathbb{P}}\left[(H - z)^+(1 - \tilde{\varphi}(z))\right], & \text{for } z < z^*, \\ z, & \text{for } z \geq z^*, \end{cases}$$

on interval $z < z^*$, and z^* is a real root of equation

$$\tilde{V}_0 = \mathbb{E}_{\mathbb{P}^*}[(H - z^*)^+].$$

Besides, one always has

$$\mathrm{CVaR}_\alpha(\hat{V}_0, \hat{\xi}) = c(\hat{z}).$$

- The dual problem is to minimize initial capital subject to a CVaR constraint:

$$\begin{cases} V_0 \longrightarrow \min\limits_{(V_0,\xi)\in\mathcal{A}}, \\ \mathrm{CVaR}_\alpha(V_0,\xi) \le \tilde{C}. \end{cases} \iff \begin{cases} \mathbb{E}_{P^*}[V_T] \longrightarrow \min\limits_{V_T\in\mathcal{F}_T}, \\ \mathrm{CVaR}_\alpha(V_T) \le \tilde{C}. \end{cases}$$

- Recall that

$$\mathrm{CVaR}_\alpha(V_0,\xi) = \min_{z\in\mathbb{R}} \left(z + \frac{1}{\alpha} \cdot \mathbb{E}_{\mathbb{P}}(H - V_T - z)^+ \right)$$

and consider a family of problems

$$\begin{cases} \mathbb{E}_{\mathbb{P}^*}[V_T] \longrightarrow \min\limits_{V_T\in\mathcal{F}_T}, \\ \mathbb{E}_{\mathbb{P}}(H - V_T - z)^+ \le (\tilde{C} - z)\cdot\alpha. \end{cases}$$

Lemma

Let $\tilde{x}$ be a solution of

$$\begin{cases} f(x) \longrightarrow \min\limits_{x \in \mathbb{X}}, \\ \min\limits_{z \in \mathbb{R}} g(x,z) \leq c. \end{cases}$$

Then the following family of problems also admits solutions, denoted $\tilde{x}(z)$:

$$\begin{cases} f(x) \longrightarrow \min\limits_{x \in \mathbb{X}}, \\ g(x,z) \leq c. \end{cases}$$

Besides, one always has

$$\tilde{x} = \tilde{x}(\tilde{z}),$$

where z is a point of global minimum of $f(\tilde{x}(z))$.

- Let $\tilde{V}_T(z)$ be the solution of

$$\begin{cases} \mathbb{E}_{\mathbb{P}^*}[V_T] \longrightarrow \min\limits_{V_T \in \mathcal{F}_T}, \\ \mathbb{E}_{\mathbb{P}}(H - V_T - z)^+ \leq (\tilde{C} - z) \cdot \alpha. \end{cases}$$

- Then the solution of the dual problem

$$\begin{cases} \mathbb{E}_{P^*}[V_T] \longrightarrow \min\limits_{V_T \in \mathcal{F}_T}, \\ \mathrm{CVaR}_\alpha(V_T) \leq \tilde{C}. \end{cases}$$

can be expressed as $\tilde{V}_T = \tilde{V}_T(\tilde{z})$,where

$$\mathbb{E}_{\mathbb{P}^*}[\tilde{V}_T(\tilde{z})] = \min_{z \in \mathbb{R}} \mathbb{E}_{\mathbb{P}^*}[\tilde{V}_T(z)].$$

If $\mathbb{E}_{\mathbb{P}}[H] > \tilde{C}\alpha$ and $\mathbb{E}_{\mathbb{P}}[(H-\tilde{C})^+] > 0$, the optimal strategy $(\hat{V}_0, \hat{\xi})$ for the dual problem is a perfect hedge for $(H-\hat{z})^+(1-\tilde{\varphi}(\hat{z}))$, where $\tilde{\varphi}(z)$ is defined by

$$
\begin{aligned}
\tilde{\varphi}(z) &= \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z)\right\}} + \gamma(z) \cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z)\right\}}, \\
\tilde{a}(z) &= \inf \left\{ a \geq 0 : \mathbb{E}_{\mathbb{P}} \left[(H-z)^+ \cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > a\right\}} \right] \leq (\tilde{C}-z)\alpha \right\}, \\
\gamma(z) &= \frac{(\tilde{C}-z)\alpha - \mathbb{E}_{\mathbb{P}} \left[(H-z)^+ \cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z)\right\}} \right]}{\mathbb{E}_{\mathbb{P}} \left[(H-z)^+ \cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z)\right\}} \right]},
\end{aligned}
$$

and $\hat{z}$ is a point of minimum of function

$$d(z) = \mathbb{E}_{\mathbb{P}^*} \left[(H-z)^+ (1-\tilde{\varphi}(z)) \right]$$

on interval $-\infty < z \leq \tilde{C}$.

Minimizing Hedging Costs
Dual Problem: Final Results (Part 2)

- If $\mathbb{E}_{\mathbb{P}}[H] \leq \tilde{C}\alpha$ or $\mathbb{E}_{\mathbb{P}}[(H - \tilde{C})^{+}] \leq 0$, the optimal strategy $(\hat{V}_0, \hat{\tilde{\xi}})$ for the dual problem is a passive strategy (do nothing).
- If the first inequality is not satisfied, target CVaR is too high compared to the expected payoff on the contingent claim, so there is no need to hedge.
- If the second inequality is not satisfied, the payoff is bounded from above by a constant less than $\tilde{C}$, so CVaR can never reach its target value no matter what strategy is used.

CVaR Hedging in the Black-Scholes Model

The Discounted Price Process

- Let the underlying S_t and bond price B_t follow

$$\begin{cases} B_t = e^{rt}, \\ S_t = S_0 \exp(\sigma W_t + \mu t). \end{cases}$$

- SDE for the discounted price process $X_t = B_t^{-1} S_t$:

$$\begin{cases} dX_t = X_t(\sigma dW_t + m dt), \\ X_0 = x_0, \end{cases}$$

$$\text{where } m = \mu - r + \frac{\sigma^2}{2}.$$

- Terminal value and Radon-Nikodym derivative:

$$\begin{aligned} X_T &= x_0 \exp\left(\sigma W_T + (m - \tfrac{1}{2}\sigma^2) T\right), \\ \frac{d\mathbb{P}^*}{d\mathbb{P}} &= \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) = \mathrm{const} \cdot X_T^{-m/\sigma^2}. \end{aligned}$$

- The contingent claim of interest is a plain vanilla call option with payoff $(S_T - K)^+$.
- The discounted payoff H is equal to

$$H = (X_T - Ke^{-rT})^+.$$

- The initial capital H_0 required for a perfect hedge is

$$H_0 = \mathbb{E}_{\mathbb{P}^*}[H] = x_0 \Phi_+(Ke^{-rT}) - Ke^{-rT} \Phi_-(Ke^{-rT}),$$

where

$$\Phi_\pm(K) = \Phi\left(\frac{\ln x_0 - \ln K}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T} \right),$$

and $\Phi(\cdot)$ is a c.d.f. for standard normal distribution.

- Assume the initial capital V_0 is limited by $\tilde{V}_0 < H_0$.
- For simplicity of presentation, assume $m > 0$.
- Our goal is to derive a hedging strategy that minimizes CVaR of the portfolio.

The optimal strategy $(\hat{V}_0, \hat{\xi})$ is a perfect hedge for $\tilde{H}(\hat{z}) = (X_T - (Ke^{-rT} + \hat{z}))^+ \cdot \mathbf{1}_{\{X_T > \tilde{b}(\hat{z})\}}$, where $\hat{z}$ is a point of global minimum of $c(z)$ on $(-\infty, z^*)$,

$$\begin{aligned} c(z) &= z + \frac{1}{\alpha} \cdot x_0 e \left[{}^{mT} \tilde{\Phi}_\pm \left(Ke^{-rT} + z \right) - \tilde{\Phi}_\pm(\tilde{b}(z)) \right] - \\ & \quad (Ke^{-rT} + z) \left[\tilde{\Phi}_\pm \left(Ke^{-rT} + z \right) - \tilde{\Phi}_\pm(\tilde{b}(z)) \right], \end{aligned}$$

where $\tilde{\Phi}_\pm(x) = \Phi_\pm\left(xe^{-mT}\right)$, z^* is the solution of

$$\tilde{V}_0 = x_0 \Phi_+(Ke^{-rT} + z^*) - (Ke^{-rT} + z^*)\Phi_-(Ke^{-rT} + z^*),$$

and for each $z \in \mathbb{R}$, $\tilde{b}(z)$ is the solution of

$$\begin{cases} x_0 \Phi_+(b) - ((Ke^{-rT} + z))\Phi_-(b) = \tilde{V}_0, \\ b \geq (Ke^{-rT} + z). \end{cases}$$

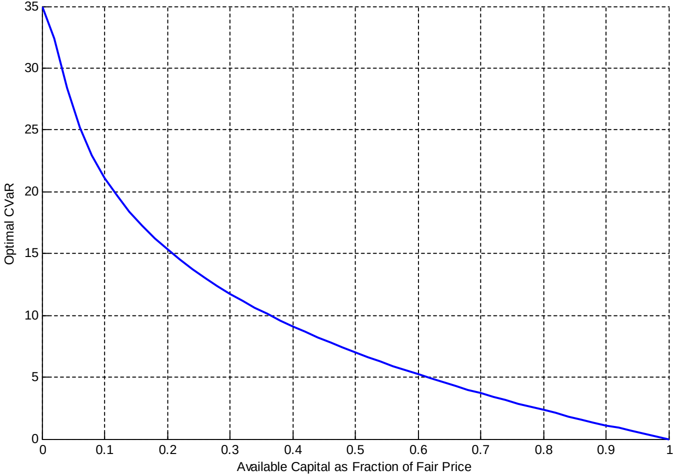
- Consider a plain vanilla call option with strike price of $K = 110$ and time to maturity $T = 0.25$.
- Assume that financial market evolves according to the Black-Scholes model with parameters

$$\sigma = 0.3, \quad \mu = 0.09, \quad r = 0.05.$$

- Initial stock price is $S_0 = 100$.
- The objective is to construct $\text{CVaR}_{0.025}$-optimal partial hedging strategies for the call option with variable amount of initial capital available, ranging from 0 to the fair price of the option.

CVaR Hedging in the Black-Scholes Model

Numerical Example: Optimal CVaR vs. Initial Capital (2)



- $(\Omega, \mathcal{F}, \mathbb{P})$ is "financial" probability space, as described earlier.
- Consider "actuarial" probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.
- Let random variable $T(x)$ denote the remaining lifetime of a person aged x.
- Let ${}_Tp_x = \tilde{\mathbb{P}}[T(x) > T]$ be a survival probability for the next T years of the insured.
- Assume that $T(x)$ does not depend on the evolution of financial market.

- Insurance company is obliged to pay the benefit in the amount of $\bar{H}$ to the insured, giving the insured is alive at time T.
- $\bar{H}$ is an $\mathcal{F}_T$-measurable non-negative random variable.
- The optimal price is traditionally calculated as an expected present value of cash flows under the risk-neutral probability.
- The "insurance" part of the contract doesn't need to be risk-adjusted since the mortality risk is unsystematic.
- Brennan-Shwartz price of the contract:

$$ {}_T U_x = \mathbb{E}_{\tilde{\mathbb{P}}} \left\{ \mathbb{E}_{\mathbb{P}^*} \left[H \cdot \mathbf{1}_{\{T(x)>T\}} \right] \right\} = {}_T p_x \cdot \mathbb{E}_{\mathbb{P}^*} \left[H \right], $$

where $H = \bar{H} e^{-rT}$ is the discounted benefit.

- The problem of the insurance company is to mitigate financial part of risk and hedge $\bar{H}$ in the financial market.
- However,

$$ {}_T U_x < \mathbb{E}_{\mathbb{P}^*}\left[H\right], $$

  hence the perfect hedge is not accessible.
- For a fixed client age x and time horizon T, denote $\tilde{V}_0 = {}_T p_x \cdot \mathbb{E}_{\mathbb{P}^*}\left[H\right]$.
- We can now consider the problem of CVaR-optimal hedging of $\bar{H}$ with capital constraint $V_0 \leq \tilde{V}_0$ and apply all techniques described earlier to derive the solution.
- The related dual problem can also be considered.

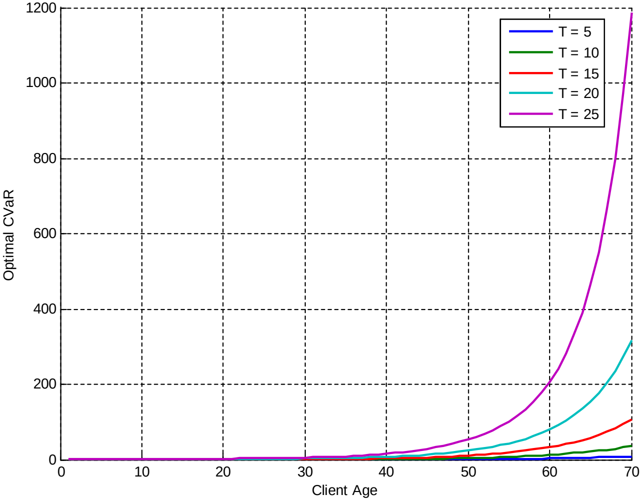
- Consider an equity-linked pure endowment contract with benefit being a call option wih strike price of $K = 110$.
- Let the starting price of the underlying be equal to $X_0 = 100$.
- Let "financial" world be driven by the Black-Scholes model:

$$\mu = 0.09, \quad r = 0.05, \quad \sigma = 0.3.$$

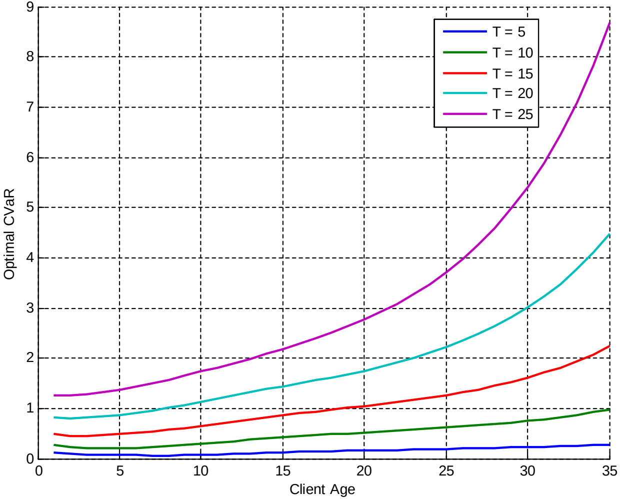
- We optimize CVaR of hedging strategies for confidence level $\alpha = 0.025$ and variable time horizon.
- We use survival probabilities from mortality table UP94 @ 2015 (Uninsured Pensioner Mortality 1994 Table Projected to the Year 2015) from McGill et al., "Fundamentals of Private Pensions" (2004)).

CVaR Hedging of Equity-Linked Insurance Contracts

Numerical Example: Optimal CVaR for Ages 1-70



Numerical Example: Optimal CVaR for Ages 1-35



1. Let $\sigma(t) \in \{1, 2\}, \sigma(0) = 1$ be a continuous time Markov chain process with Markov generator
$$L_\sigma = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$
2. Define the main driving factors of the market:
$$X_t = \int_0^t c_{\sigma(s)} ds, \qquad J_t = \sum_0^{N_t} h_{\sigma(T_j-)},$$
where $\mathbf{c} = (c_1, c_2)$, $\mathbf{h} = (h_1, h_2)$ and N_t is the number of jumps of $\sigma(t)$ up to time t.
3. The risk-free asset is defined by $dB_t = r_{\sigma(t)} B_t dt$, and the interest rate $\mathbf{r}$ has two states (r_1, r_2).
4. The risky asset is defined similarly to Merton's model:
$$dS_t = S_{t-} d(X_t + J_t).$$

- Telegraph market model can be described as a complete market model with two traded assets, where dynamics of the risky asset features jumps and regime switching.
- The model can be viewed as a generalization of Merton's model preserving completeness of the market.

Theorem

The telegraph model is arbitrage free if and only if

$$\frac{r_\sigma - c_\sigma}{h_\sigma} > 0, \quad \sigma = 1, 2.$$

If the model is arbitrage free, it is complete.

- Our algorithm for deriving CVaR-optimal strategies requires computing expectations of the form

$$\mathbb{E}[f(S_T, B_T) \cdot \mathbf{1}_{\{Z_T < a\}}]$$

for various functions f and constants a, both under the statistical measure $\mathbb{P}$ and under the risk-neutral measure $\mathbb{P}^*$.

- S_t, B_t and Z_t may all be expressed in terms of X_t and N_t; consider $g(\cdot, \cdot)$ such that

$$\mathbb{E}[f(S_T, B_T) \cdot \mathbf{1}_{\{Z_T < a\}}] = \mathbb{E}[g(X_t, N_t)].$$

- Expand the expected value above by conditioning on $N_t = n$:

$$\mathbb{E}[g(X_t, N_t)] = \sum_{n \geq 0} \int_{\mathbb{R}} g(x, n) p_n(t, x) dx,$$

where $p_n(t, x)$ is defined as

$$p_n(t, x) = \frac{d}{dx} \mathbb{P}\left[\{X_t < x\} \cap \{N_t = n\}\right].$$

For all $t \geq 0$ and $x \in \mathbb{R}$,

$$p_0(t,x) = e^{-\lambda_1 t}\delta(x - c_1 t)$$

and for all $k \geq 1$

$$\begin{aligned} p_{2k-1}(t,x) &= \frac{\lambda_1 \left(\phi_1(t,x)\phi_2(t,x)\right)^{k-1}}{\mid c_2 - c_1 \mid \left((k-1)!\right)^2} \exp\left(-\phi_1(t,x) - \phi_2(t,x)\right), \\ p_{2k}(t,x) &= \frac{p_{2k-1}(t,x)\phi_2(t,x)}{k}, \end{aligned}$$

where

$$\begin{aligned} \phi_1(t,x) &= \lambda_1 \frac{c_2 t - x}{c_2 - c_1}, \\ \phi_2(t,x) &= \lambda_2 \frac{x - c_1 t}{c_2 - c_1}, \end{aligned}$$

and $x \in \left(\min\{c_1 t, c_2 t\}, \max\{c_1 t, c_2 t\}\right)$.

- Consider a plain vanilla call option with strike price of $K = 100$ and time to maturity $T = 1$.
- Assume that financial market evolves according to the telegraph market model with parameters

$$\begin{aligned} c_1 &= -0.5, \quad c_2 = 0.5, \\ \lambda_1 &= \lambda_2 = 5, \\ r_1 &= r_2 = 0.07, \\ h_1 &= 0.5, \quad h_2 = -0.35. \end{aligned}$$

- Initial stock price is $S_0 = 100$.
- The objective is to construct $\text{CVaR}_{0.025}$-optimal partial hedging strategies for the call option with variable amount of initial capital available, ranging from 0 to the fair price of the option.

CVaR Hedging in Telegraph Market Model

Numerical Example: Optimal CVaR vs. Initial Capital (2)