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Efficient Pricing of CMS Spread Options in a Stochastic Volatility Libor Market Model

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Outline

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Part 1: CMS Spread Option Pricing in a SV-LMM

- General Problem
- Spread option formulas
- Numerical Results

Part 2: Extracting Correlations from the Market

- Correlation Parameterizations
- Calibration Example

Motivation

- Standard approach: Calibration of volatilities to caps/swaptions. Forward rate correlations are usually estimated from historical data.
- Swaptions exhibit only weak dependence on correlations.
- Even historical estimation of forward rate correlation matrices not trivial (choice of yield curve model, time horizon, backward-looking)
- Goal: Augment set of calibration instruments with CMS spread options.
  $\Rightarrow$ Consistent pricing of spread-related exotics.

Model framework

SV-LMM with time-dep. skews (Piterbarg, 2005)

$$
\begin{aligned}
dL_i(t) &= \ldots dt + [\beta_i(t)L_i(t) + (1-\beta_i(t))L_i(0)]\sqrt{V(t)}\sigma_i(t)dW_i(t),\\
dV(t) &= \kappa(1-V(t))dt + \xi\sqrt{V(t)}dZ(t)
\end{aligned}
$$

$$
\begin{aligned}
\langle dW_n(t), dW_m(t)\rangle &= \rho(t, T_n, T_m)dt\\
\langle dW_n(t), dZ(t)\rangle &= 0
\end{aligned}
$$

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Model framework

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Swap-rate dynamics

For the swap rates

$$S_j = \sum w_i L_i$$

we have approximately

$$dS_j(t) = \dots dt + [\beta_j(t)S_j(t) + (1-\beta_j(t))S_j(0)]\sqrt{V(t)}\sigma_j(t)dW_j(t),$$

i.e., they are again of type 'displaced Heston'.

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Problem to be solved

For the calibration to CMS spread options we must be able to rapidly calculate expectations of the form

$$\mathbb{E}[(S_1(T) - S_2(T) - K)^+],$$

where S_1 and S_2 are for example a 10Y and a 2Y swap rate, respectively.

$\Rightarrow$ No Monte-Carlo !

Method I: Antonov & Arneguy (2009), Hurd & Zhou (2009)

Derive the 2-dim Fourier transform $\hat{F}$ of the payoff function

$$F(x_1, x_2) = (e^{x_1} - e^{x_2} - 1)^+$$

via complex Γ-function.

$\Rightarrow$ Price of spread option given as 2-dim inverse Fourier transform

$$\mathsf{SprOpt}(0) = \frac{1}{2\pi} \iint_{\mathbb{R}^2 + i\varepsilon} e^{iuX_0'} \hat{F}(u_1, u_2) \Phi(u_1, u_2) d(u_1, u_2),$$

where Φ denotes char. function of $(\log S_1(T), \log S_2(T))$.

However: Efficient 2-dim Fourier inversion not trivial ! (Choice of ε, highly oscillatory integrand for short maturities)

Method II: Kiesel & Lutz (2010)

Brownian motion driving stochastic volatility is independent of the swap rate Brownian motions !

Define the integrated variance

$$\overline{V}_T := \int_0^T V_t dt.$$

$$\Rightarrow \; (S_1(T), S_2(T)) \,|\, \overline{V}_T \quad \text{jointly log-normal (if skews} = 1).$$

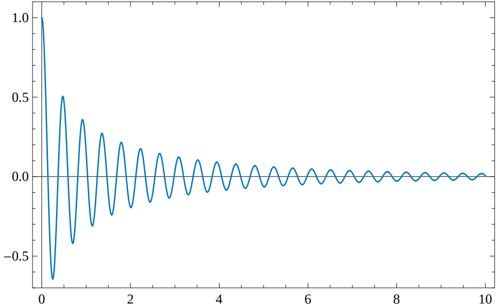
$$\Rightarrow \mathbb{E}[(S_1(T) - S_2(T) - K)^+] = \mathbb{E}\Big[\mathbb{E}[(S_1(T) - S_2(T) - K)^+ | \bar{V}_T]\Big]$$
$$= \int_0^\infty \int_{-\infty}^\infty g(u,v) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du f(v) dv,$$

where $g(u,v)$ is some 'nice' real function and f the density of the integrated variance $\overline{V}_T$.

Inverse transform

Problem: Density f of $\overline{V}_T$ only given via Fourier inversion $\rightsquigarrow$ a further integral !

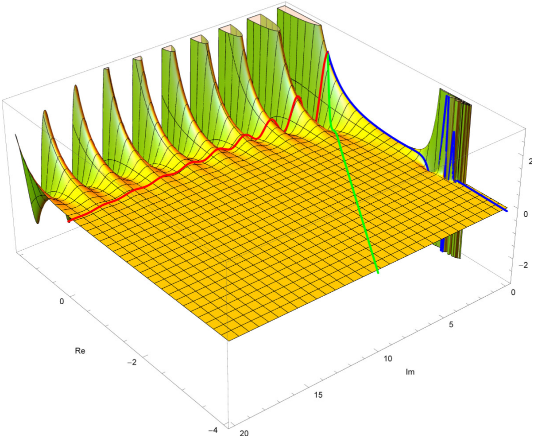
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{x(a+iu)} \hat{f}(a+iu) du.$$



$\Rightarrow$ For each x at which the density is to be evaluated we need to calculate an oscillatory integral.

Inverse transform II

Real part of the integrand in the complex plane

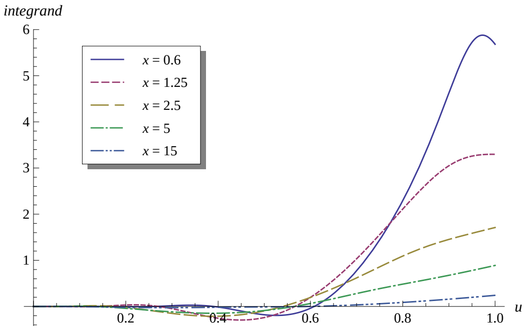


Green line: 'optimal' linear contour.

Inverse transform III

With optimal linear contour and after transformation to finite integration interval we have

$$f(x) = -\frac{1}{\pi}\int_0^1 \mathrm{Im}\left(e^{x\tilde{s}(u)}\hat{f}(\tilde{s}(u))\tilde{s}'(u)\right) du$$



⇒ With an adaptive integration scheme usually 20-40 sampling points are sufficient for most practical applications.

Remarks

We can now rapidly calculate spread option prices for two correlated displaced Heston processes.

For application within LMM we further need

- appropriate approximations for swap rate drift terms (convexity adjustments)
- replace time-dep. swap-rate parameters with constants via parameter averaging (see Piterbarg (2005)).

$\rightarrow$ see Kiesel & Lutz 2010.

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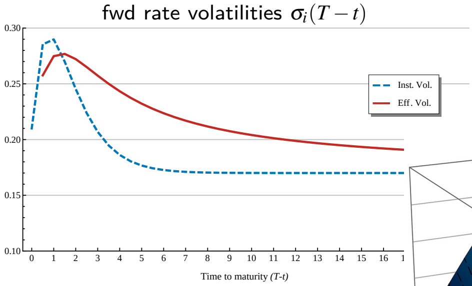
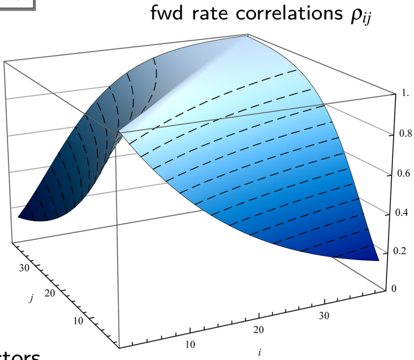
Numerical Results





- 20Y LMM based on 6M rates, 5 factors
- Typical upward sloping initial yield curve (from 3% to 5%)
- Skew parameters: linearly decreasing from 90% to 40%
- SV parameters: $\kappa = 15\%$, $\xi = 130\%$

Numerical Results

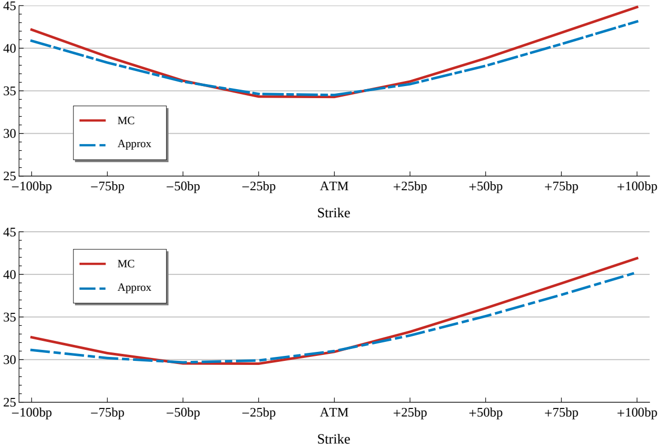


Figure: Implied normal spread volatilities in basis points for 10Y-2Y CMS spread options with 5Y maturity (top) and 10Y maturity (bottom). Computing time $\sim$ 40 ms.

Outline

Correlation parameterizations

- **Schoenmakers-Coffey (2003) 2-parametric form:**

$$\rho_{ij} = \exp\left(-\frac{|i-j|}{N-1} \left(-\log \rho_\infty + \eta\, h(i,j) \right) \right),$$
$$\rho_\infty \in (0,1),\ \eta \in [0, -\log \rho_\infty]$$

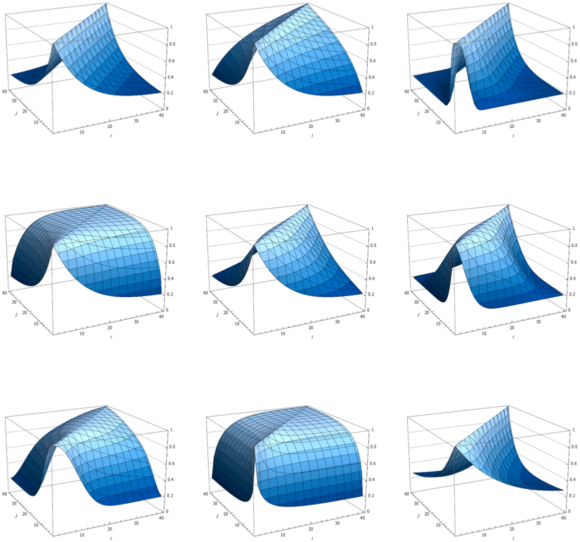
- **Rebonato (2004) 3-parametric form (not necessarily pos. def.)**

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty) \exp\left(-\beta |i-j| \exp\{-\alpha \min(i,j)\} \right),$$
$$\beta > 0,\ \alpha \in \mathbb{R},\ \rho_\infty \in [0,1)$$

- **Lutz (2010) 5-parametric form:**

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty) \left[\exp\left(-\beta (i^\alpha + j^\alpha)\right) + \vartheta_{ij}(\delta,\gamma) \sqrt{\left(1-\exp\{-2\beta i^\alpha\}\right)\left(1-\exp\{-2\beta j^\alpha\}\right)} \right]$$
$$\alpha, \beta > 0, \quad \gamma, \delta \in \mathbb{R}, \quad \rho_\infty \in [0,1).$$

Lutz (2010) 5-parametric form



Calibration example

- 30Y semi-annual LMM, 10 factors.
- Piecewise const. forward rate volatility and skew functions.
- EUR market data as of 01/14/2008.
- Calibration targets: 9 caplet smiles, 36 swaption smiles, 7 CMS spread option (10Y-2Y) implied volatilities.
- Computing time: 1:54 min.

	RMSE	
	ATM	Smile
caplets (Black-vol)	0.24%	0.48%
swaptions (Black-vol)	0.25%	0.31%
	$K = 0.25\%$	Smile
CMSSOs (bp vol)	0.7	2.4

Calibration example

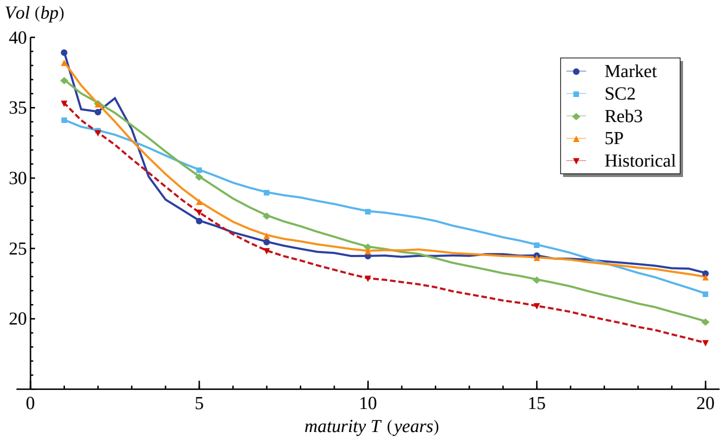


Figure: 10Y-2Y CMS spread option implied volatilities for $K = 0.25\%$.

Calibration example

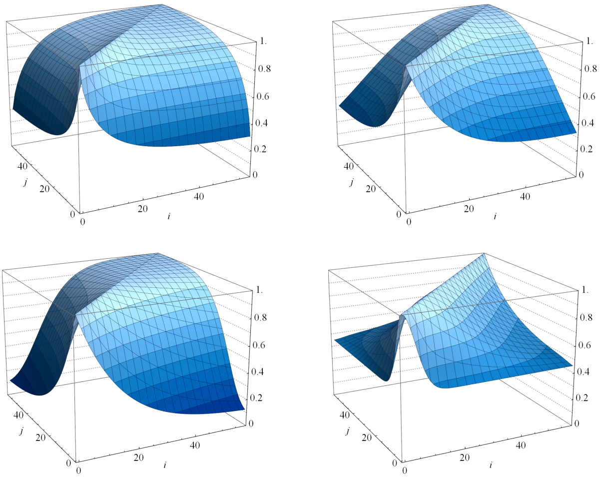


Figure: Calibrated correlation matrices. From top left to bottom right: 5P fitted to historical matrix, SC2, Reb3, 5P.

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Thank you for your attention !