Robust Pricing and Hedging of Options on Variance

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Financial Setting

- Option priced on an underlying asset S_t
- Dynamics of S_t unspecified, but suppose paths are continuous, and we see prices of call options at all strikes K and at maturity time T
- Assume for simplicity that all prices are discounted — this won't affect our main results
- Under risk-neutral measure, S_t should be a (local-)martingale, and we can recover the law of S_T at time T from call prices $C(K)$. (Breeden-Litzenberger)

Financial Setting

- Given these constraints, what can we say about market prices of other options?
- Two questions:
  - What prices are consistent with a model?
  - If there is no model, is there an arbitrage which works for every model in our class — robust!
- Intuitively, understanding ‘worst-case’ model should give insight into any corresponding arbitrage.
- Insight into hedge likely to be more important than pricing
- But... prices will indicate size of ‘model-risk’

Connection to Skorokhod Embeddings

- Under a risk-neutral measure, expect S_t to be a local-martingale with known law at time T, say μ.
- Since S_t is a continuous local martingale, we can write it as a time-change of a Brownian motion: $S_t = B_{A_t}$
- Now the law of B_{A_T} is known, and A_T is a stopping time for B_t
- Correspondence between possible price processes for S_t and stopping times τ such that $B_\tau \sim \mu$.
- Problem of finding τ given μ is Skorokhod Embedding Problem
- Commonly look for ‘worst-case’ or ‘extremal’ solutions
- Surveys: Obłój, Hobson,...

Variance Options

- We may typically suppose a model for asset prices of the form:
$$\frac{dS_t}{S_t} = \sigma_t dW_t,$$
where W_t a Brownian motion.
- the volatility, σ_t, is a predictable process
- Recent market innovations have led to asset volatility becoming an object of independent interest
- For example, a variance swap pays:
$$\int_0^T \left(\sigma_t^2 - \bar{\sigma}^2\right) dt$$
where $\bar{\sigma}_t$ is the 'strike'. Dupire (1993) and Neuberger (1994) gave a simple replication strategy for such an option.

Variance Call

- A variance call is an option paying:

$$(\langle \ln S \rangle_T - K)_+$$

- Let $dX_t = X_t \, d\tilde{W}_t$ for a suitable BM $\tilde{W}_t$
- Can find a time change τ_t such that $S_t = X_{\tau_t}$, and so:

$$d\tau_t = \frac{\sigma_t^2 S_t^2}{S_t^2} \, dt$$

- And hence

$$(X_{\tau_T}, \tau_T) = \left(S_T, \int_0^T \sigma_u^2 \, du \right) = (S_T, \langle \ln S \rangle_T)$$

- More general options of the form: $F(\langle \ln S \rangle_T)$.

Variance Call

- This suggests finding lower bound on price of variance call with given call prices is equivalent to:

$$\text{minimise: } \mathbb{E}(\tau - K)_+ \text{ subject to: } \mathcal{L}(X_\tau) = \mu$$

  where μ is a given law.
- Is there a Skorokhod Embedding which does this?

Root Construction

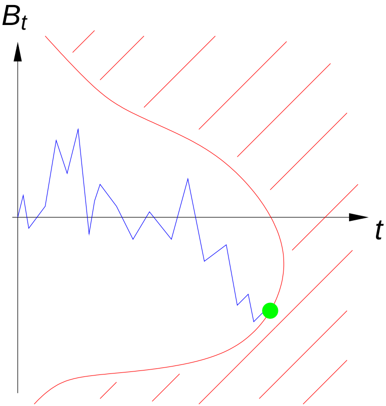
- $\beta \subseteq \mathbb{R} \times \mathbb{R}_+$ a barrier if:

  $$(x,t) \in \beta \implies (x,s) \in \beta$$

  for all $s \geq t$
- Given μ, exists β and a stopping time

  $$\tau = \inf\{t \geq 0 : (B_t, t) \in \beta\}$$

  which is an embedding.
- Minimises $\mathbb{E}(\tau - K)_+$ over all (UI) embeddings
- Construction and optimality are subject of this talk



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- Rost (1976)

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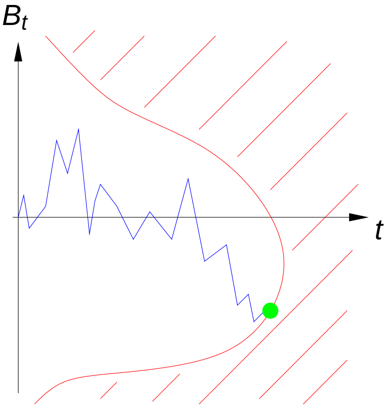

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- Root proved this for X_t a Brownian motion. Rost (1976) extended his solution to much more general processes, and proved optimality, which was conjectured by Kiefer.
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Questions

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This known connection leads to two important questions:

1. *How do we find the Root stopping time?*
2. *Is there a corresponding hedging strategy?*

- Dupire has given a connected free boundary problem
- Dupire, Carr & Lee have given strategies which sub/super-replicate the payoff, but are not necessarily optimal
- Hobson has given a formal, but not easily solved, condition a hedging strategy must satisfy.

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Root's Problem

We want to connect Root's solution and the solution of a free-boundary problem. We will consider the case where $X_t = \sigma(X_t)dB_t$ and σ is nice (smooth, Lipschitz, strictly positive on $(0, \infty)$). To make explicit the first, we define:

Root's Problem (RP)

Find an open set $D \subset \mathbb{R} \times \mathbb{R}_+$ such that $(\mathbb{R} \times \overline{\mathbb{R}_+})/D$ is a barrier generating a UI stopping time τ_D and $X_{\tau_D} \sim \mu$.

Here we denote the exit time from D as τ_D.

Free Boundary Problem

Free Boundary Problem (FBP)

To find a continuous function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and a connected open set $D : \{(x, t),\ 0 < t < R(x)\}$ where $R : \mathbb{R} \to \overline{\mathbb{R}}_+ = [0, \infty]$ is a lower semi-continuous function, and

$$u \in \mathbb{C}^0(\mathbb{R} \times [0, \infty)) \quad \text{and} \quad u \in \mathbb{C}^{2,1}(D);$$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma(x)^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{on } D; \quad u(x, 0) = -|x - S_0|;$$

$$u(x, t) = U_\mu(x) = -\int |x - y|\, \mu(dy), \quad \text{if } t \geq R(x),$$

$u(x, t)$ is concave with respect to $x \in \mathbb{R}$.

$\frac{\partial^2 u}{\partial x^2}$ ‘disappears’ on ∂D.

(RP) is equivalent to (FBP)

An easy connection is then the following:

Theorem

Under some conditions on D, if D is a solution to (RP), we can find a solution to (FBP). In addition, this solution is unique.

Sketch Proof of (RP) $\implies$ (FBP)

Simply take

$$u(x,t) = -\mathbb{E}|X_{t\wedge\tau_D} - x|.$$

Resulting properties are mostly straightforward/follow from regularity of D^C, and fact that, for $(x,t) \in D^C$:

$$-\mathbb{E}|X_{t\wedge\tau_D} - x| = -\mathbb{E}|X_{\tau_D} - x|.$$

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Optimality of Root's Barrier

Rost's Result

Given a function F which is convex, increasing, Root's barrier solves:

$$\begin{aligned} &\text{minimise} && \mathbb{E}F(\tau) \\ &\text{subject to:} && X_\tau \sim \mu \\ & && \tau \text{ a (UI) stopping time} \end{aligned}$$

Want:

- A simple proof of this. . .
- . . . that identifies a 'financially meaningful' hedging strategy.

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Optimality

Write $f(t) = F'(t)$, define

$$M(x,t) = \mathbb{E}^{(x,t)} f(\tau_D),$$

and

$$Z(x) = 2\int_0^x \int_0^y \frac{M(z,0)}{\sigma^2(z)}\,\mathrm{d}z\,\mathrm{d}y,$$

so that in particular, $Z''(x) = 2\sigma^2(x)M(x,0)$. And finally, let:

$$G(x,t) = \int_0^t M(x,s)\,ds - Z(x).$$

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Optimality

Then there are two key results:

Proposition (▸ Proof)

For all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$*:*

$$G(x, t) + \int_0^{R(x)} (f(s) - M(x, s))\,\mathrm{d}s + Z(x) \leq F(t).$$

Theorem (▸ Proof)

We have:

$$G(X_t, t) \text{ is a submartingale,}$$

and

$$G(X_{t \wedge \tau_D}, t \wedge \tau_D) \text{ is a martingale.}$$

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We can now show optimality. Recall we had:

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But $\int_0^{R(x)} (f(s) - M(x,s))\,\mathrm{d}s + Z(x)$ is just a function of x, so

$$G(X_t,t) + H(X_t) \le F(t).$$

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Hedging Strategy

Since $G(X_t, t)$ is a submartingale, there is a trading strategy which sub-replicates $G(X_t, t)$:

$$G(S_t, \langle \ln S \rangle_t) \geq \int_0^t \frac{G_x(S_r, \langle \ln S \rangle_r)}{\sigma_r^2} \, dS_r$$

and $H(X_t)$ can be replicated using the traded calls; moreover, in the case where $\tau = \tau_D$, we get equality, so this is the best we can do.

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Numerical implementation

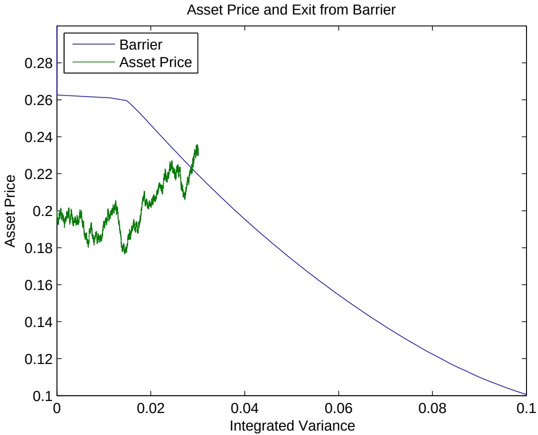
- How ‘good’ is the subhedge in practice?
- Take an underlying Heston process:

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}dB_t^1$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dB_t^2$$

  where B_t^1, B_t^2 are correlated Brownian motions, correlation ρ.
- Compute Barrier and hedging strategies based on the corresponding call prices.
- How does the subhedging strategy behave under the ‘true’ model?
- How does the strategy perform under another model?

Numerical Implementation

- Payoff: $\frac{1}{2}\left(\int_0^T \sigma_t \, dt\right)^2$. Parameters: $T = 1, r = 0.05, S_0 = 0.2, \sigma_0^2 = 0.4, \kappa = 10, \theta = 0.4, \xi = 1.0, \rho = -1.0$. Prices: actual 9.80×10^{-4}, subhedge 5.463×10^{-4}.



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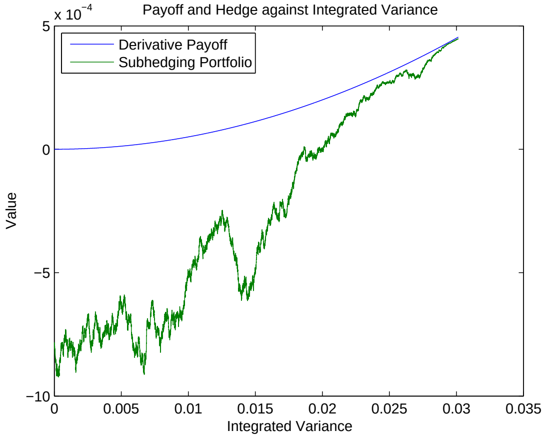

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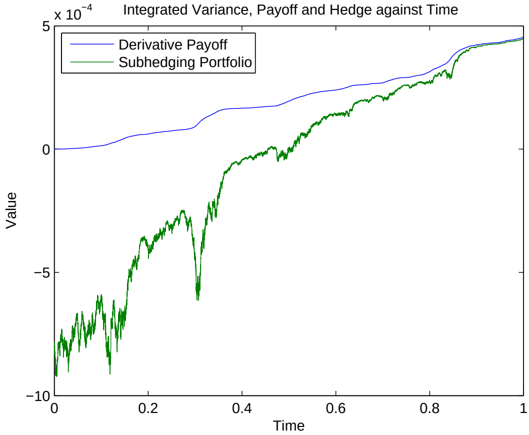

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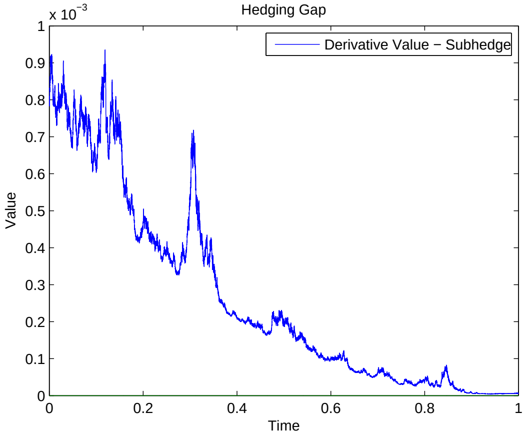

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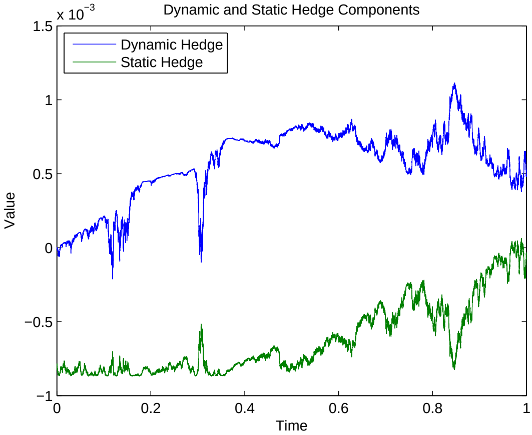

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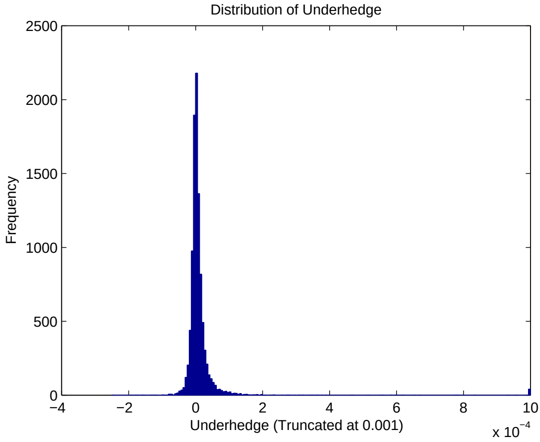

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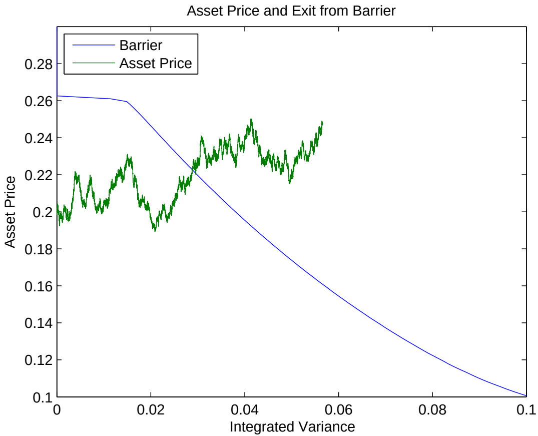

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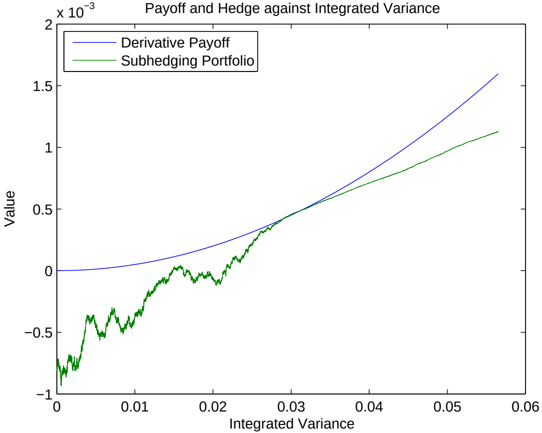

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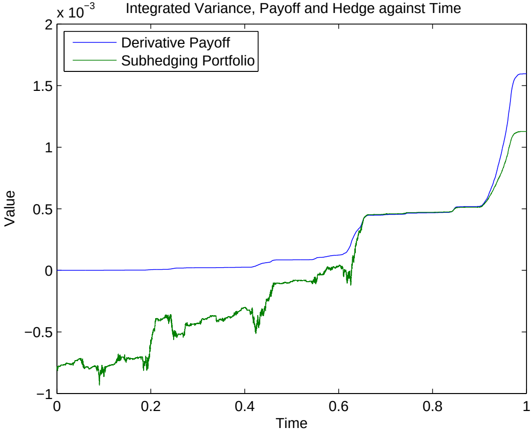

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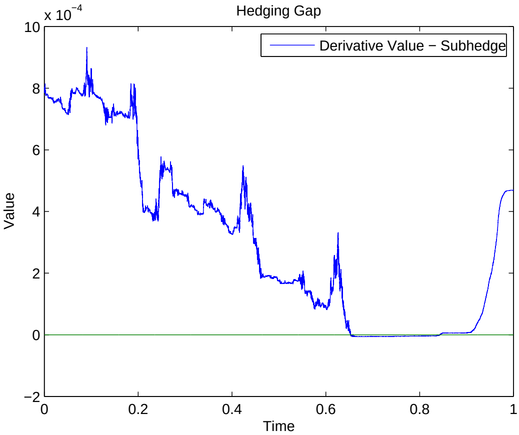

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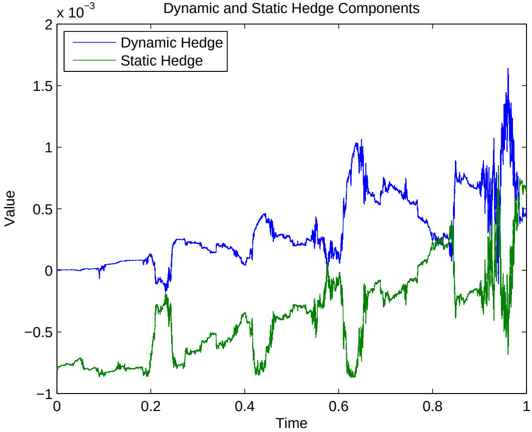

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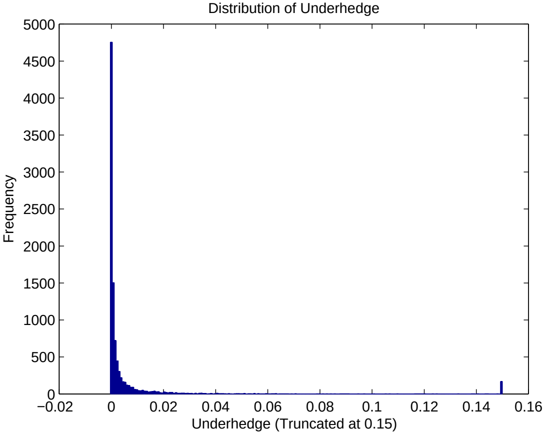

Numerical Implementation: ‘Variance Call’

- Payoff: $\left(\int_0^T \sigma_t^2\, dt - K\right)_+$. Prices: actual $= 0.0106$, subhedge $= 0.0076$.
- Parameters: $T = 1$, $r = 0.05$, $S_0 = 0.2$, $\sigma_0^2 = 0.0174$, $\kappa = 1.3253$, $\theta = 0.0354$, $\xi = 0.3877$, $\rho = -0.7165$, $K = 0.02$.

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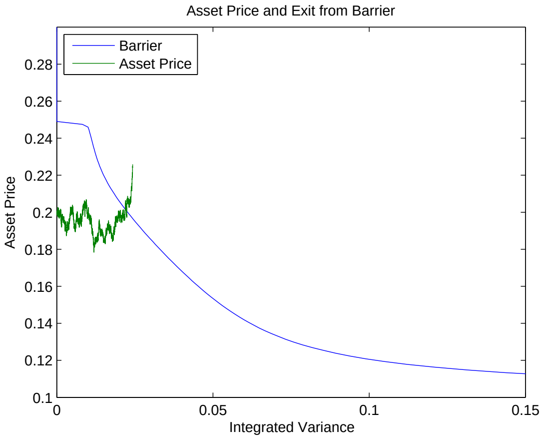

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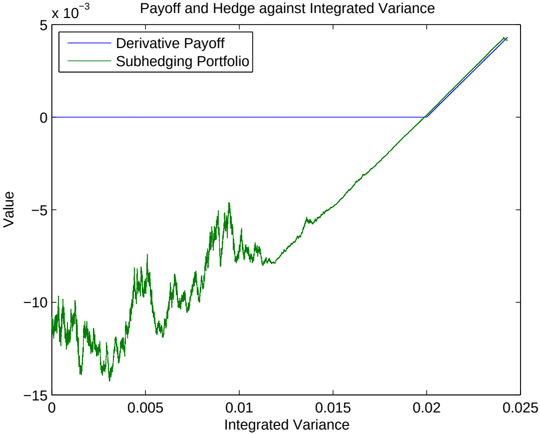

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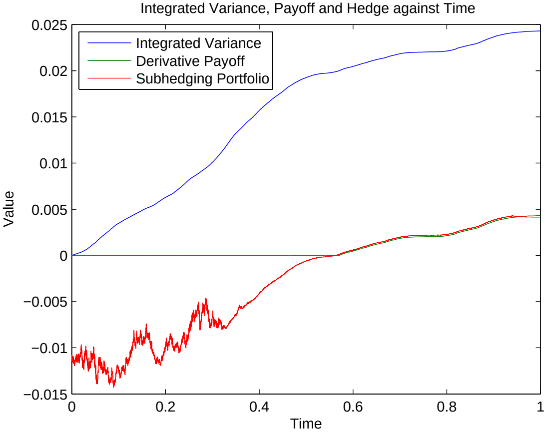

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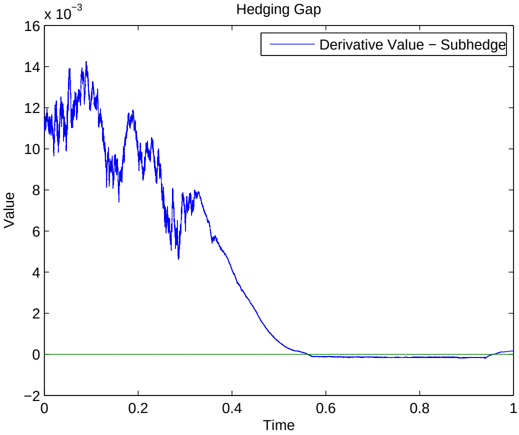

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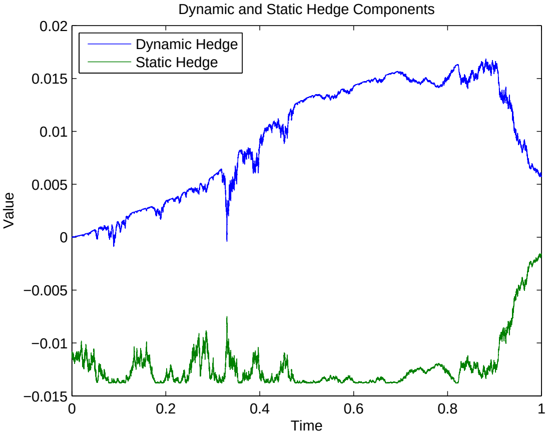

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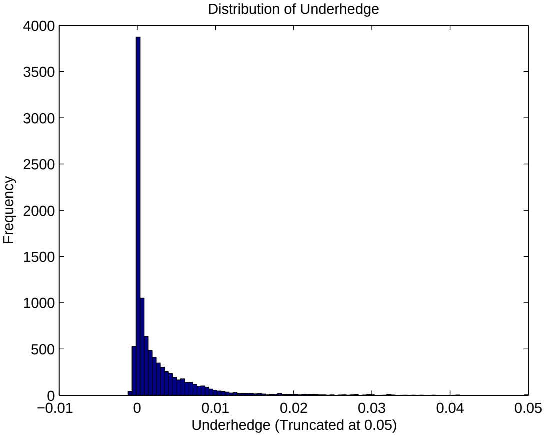

Conclusion

- Lower bounds on Pricing Variance options $\sim$ finding Root's barrier
- Equivalence between Root's Barrier and a Free Boundary Problem
- New proof of optimality, which allows explicit construction of a pathwise inequality
- Financial Interpretation: model-free sub-hedges for variance options.

Proof of Proposition

If $t \leq R(x)$ then the left-hand side is:

$$\int_0^t f(s)\,\mathrm{d}s - \int_t^{R(x)} M(x,s)\,\mathrm{d}s = F(t) - \int_t^{R(x)} M(x,s)\,\mathrm{d}s$$

And $M(x,s) \geq f(s) \geq 0$.

If $t \geq R(x)$, we get:

$$\int_{R(x)}^t M(x,s)\,\mathrm{d}s + \int_0^{R(x)} f(s)\,\mathrm{d}s = \int_{R(x)}^t f(s)\,\mathrm{d}s + \int_0^{R(x)} f(s)\,\mathrm{d}s$$
$$= F(t).$$

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$\square$

Optimality

Recalling that $M(x,t) = \mathbb{E}^{(x,t)} f(\tau_D)$, we have:

$$\mathbb{E}\left[M(X_t,u)|\mathcal{F}_s\right] \geq \begin{cases} M(X_s, s-t+u) & u \geq t-s \\ \mathbb{E}\left[M(X_{t-u},0)|\mathcal{F}_s\right] & u \leq t-s \end{cases}.$$

And by Itô:

$$\mathbb{E}\left[Z(X_t) - Z(X_s)|\mathcal{F}_s\right] = \int_s^t M(X_r,0)\,dr, \quad s \leq t.$$

Then it can be shown:

$$\mathbb{E}[G(X_t,t)|\mathcal{F}_s] \geq G(X_s,s).$$

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Proof of Submartingale Condition

$$\mathbb{E}\left[G(X_t,t)|\mathcal{F}_s\right] = \int_0^t \mathbb{E}\left[M(X_t,u)|\mathcal{F}_s\right] du - \mathbb{E}\left[Z(X_t)|\mathcal{F}_s\right]$$

$$= G(X_s,s) + \int_0^t \mathbb{E}\left[M(X_t,u)|\mathcal{F}_s\right] du$$

$$- \int_0^s M(X_s,u)\,du - \mathbb{E}\left[Z(X_t) - Z(X_s)|\mathcal{F}_s\right]$$

$$\geq G(X_s,s) + \int_0^{t-s} \mathbb{E}\left[M(X_{t-u},0)|\mathcal{F}_s\right] du$$

$$- \int_0^s M(X_s,u)\,du - \int_s^t \mathbb{E}\left[M(X_u,0)|\mathcal{F}_s\right] du$$

$$+ \int_{t-s}^t M(X_s,s-t+u)\,du$$

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&= G(X_s,s) + \int_0^t \mathbb{E}\left[M(X_t,u)|\mathcal{F}_s\right] du \\
&\quad - \int_0^s M(X_s,u)\,du - \mathbb{E}\left[Z(X_t) - Z(X_s)|\mathcal{F}_s\right] \\
&\geq G(X_s,s) + \int_0^{t-s} \mathbb{E}\left[M(X_{t-u},0)|\mathcal{F}_s\right] du \\
&\quad - \int_0^s M(X_s,u)\,du - \int_s^t \mathbb{E}\left[M(X_u,0)|\mathcal{F}_s\right] du \\
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&\geq G(X_s,s) + \int_0^{t-s} \mathbb{E}\left[M(X_{t-u},0)|\mathcal{F}_s\right]du \\
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Proof of Submartingale Condition

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A somewhat similar computation gives:

$$\mathbb{E}\left[G(X_{t\wedge\tau_D}, t\wedge\tau_D)|\mathcal{F}_s\right] = G(X_s,s)$$

on $\{s \leq \tau_D\}$.

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& - \int_s^t \mathbb{E}\left[M(X_u,0)|\mathcal{F}_s\right] du + \int_0^s M(X_s,u)\, du \\
& - \int_0^s M(X_s,u)\, du \\
\geq\ & G(X_s,s).
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